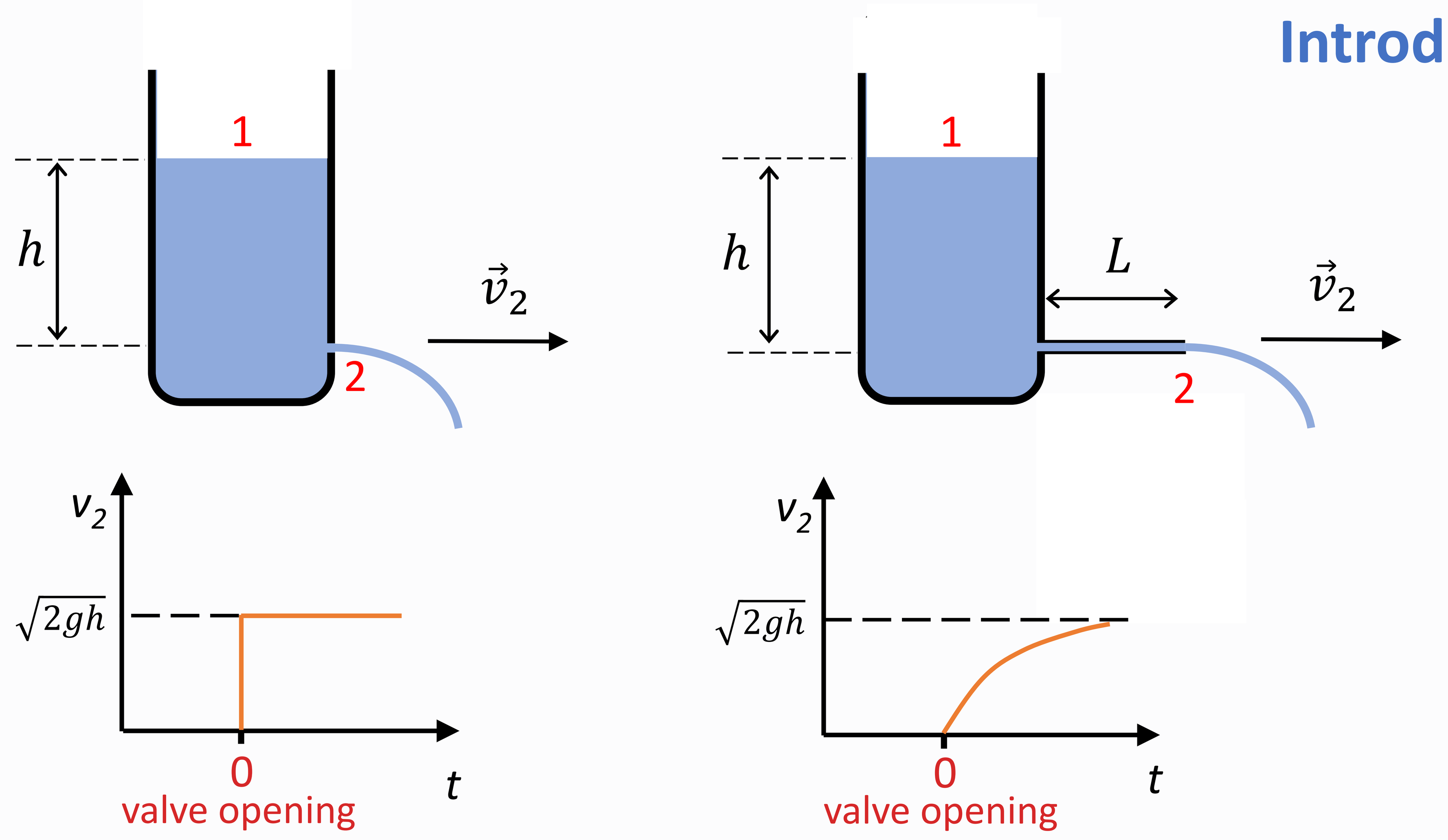
Testing Unsteady Bernoulli’s Equation: an Undergraduate Laboratory Project

UNIVERSITÀ DI PAVIA
Dipartimento di Fisica "Alessandro Volta"

Enrico Giulotto, Massimo Bertocchi, and Massimiliano Malgieri
Department of Physics A. Volta, University of Pavia, via Bassi 6, 27100 Pavia, Italy

Introduction



Discharge of an ideal fluid from a pipe connected to a large tank can be predicted by unsteady Bernoulli’s equation:

$$\left(p+\frac{1}{2}\rho v^2+\rho g z\right)_1=\left(p+\frac{1}{2}\rho v^2+\rho g z\right)_2+\int_1^2 \rho\frac{\partial v}{\partial t}ds$$

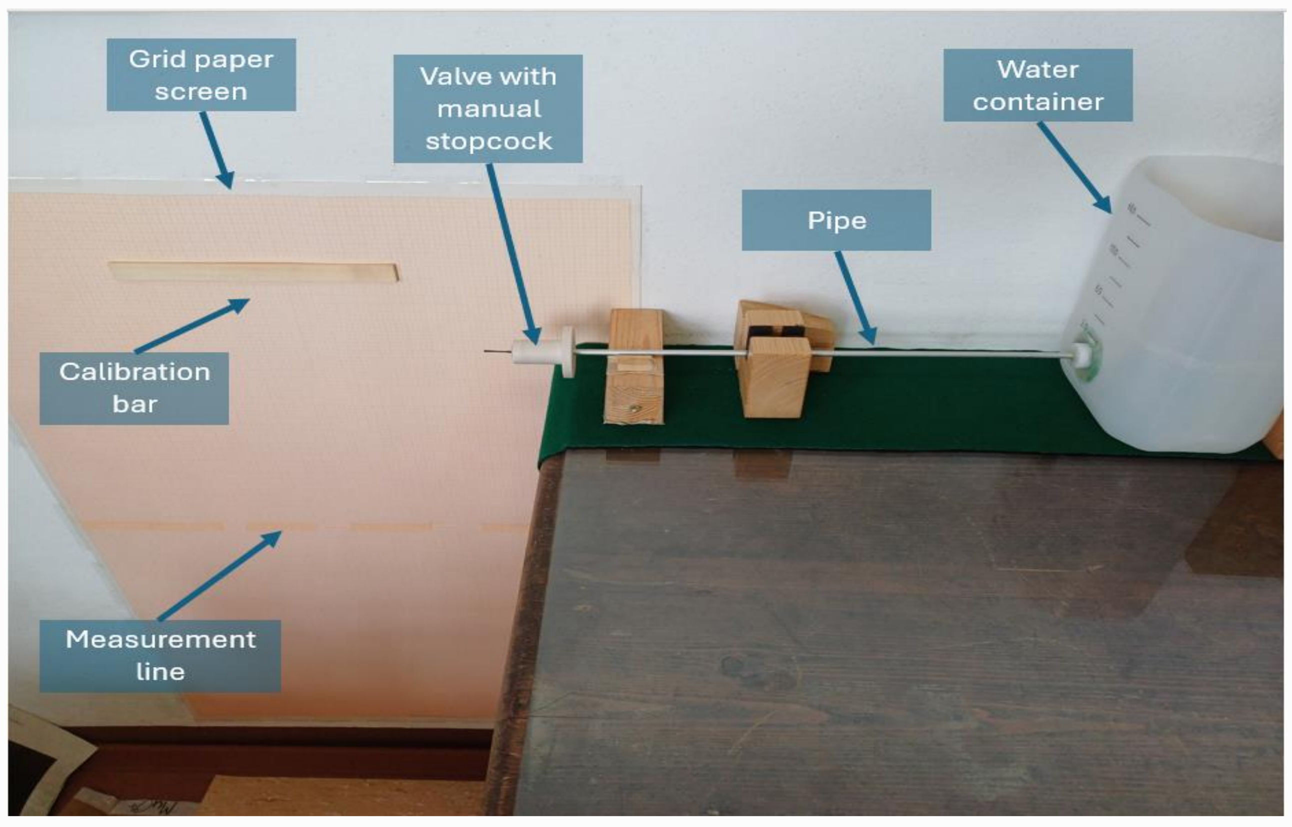
The result can be approximated to: $v_2(t)=\sqrt{2gh}\,\tanh\left(\sqrt{\frac{gh}{2}}\frac{t}{L}\right)$

The velocity in the pipe v_2 approaches $\sqrt{2gh}$ with a characteristic time constant: $\tau = L\sqrt{2/gh}$

For $L \longrightarrow 0$, the time constant τ vanishes and the steady velocity is $v_s=\sqrt{2gh}$, in agreement with the well-known Torricelli’s law.

We propose low-cost apparatus (tank with a pipe) and procedure to test the unsteady Bernoulli’s equation.

Experimental details



- The apparatus was realized by using low-cost or recycled materials.
- Rapid opening of a magnetic valve allowed the spillage of the water.
- The 60 FPS videos showing the time evolution of the jet exiting the pipe were recorded for different values of h, from 4 to 14 cm in steps of 2cm.
- The smartphone camera was equipped with a 50 mm equivalent lens.
- Length and inner diameter of the pipe were 470 mm and 3.9 mm, respectively.
- A few drops of food coloring were added to the water to improve the visibility of the jet.

Data analysis and results

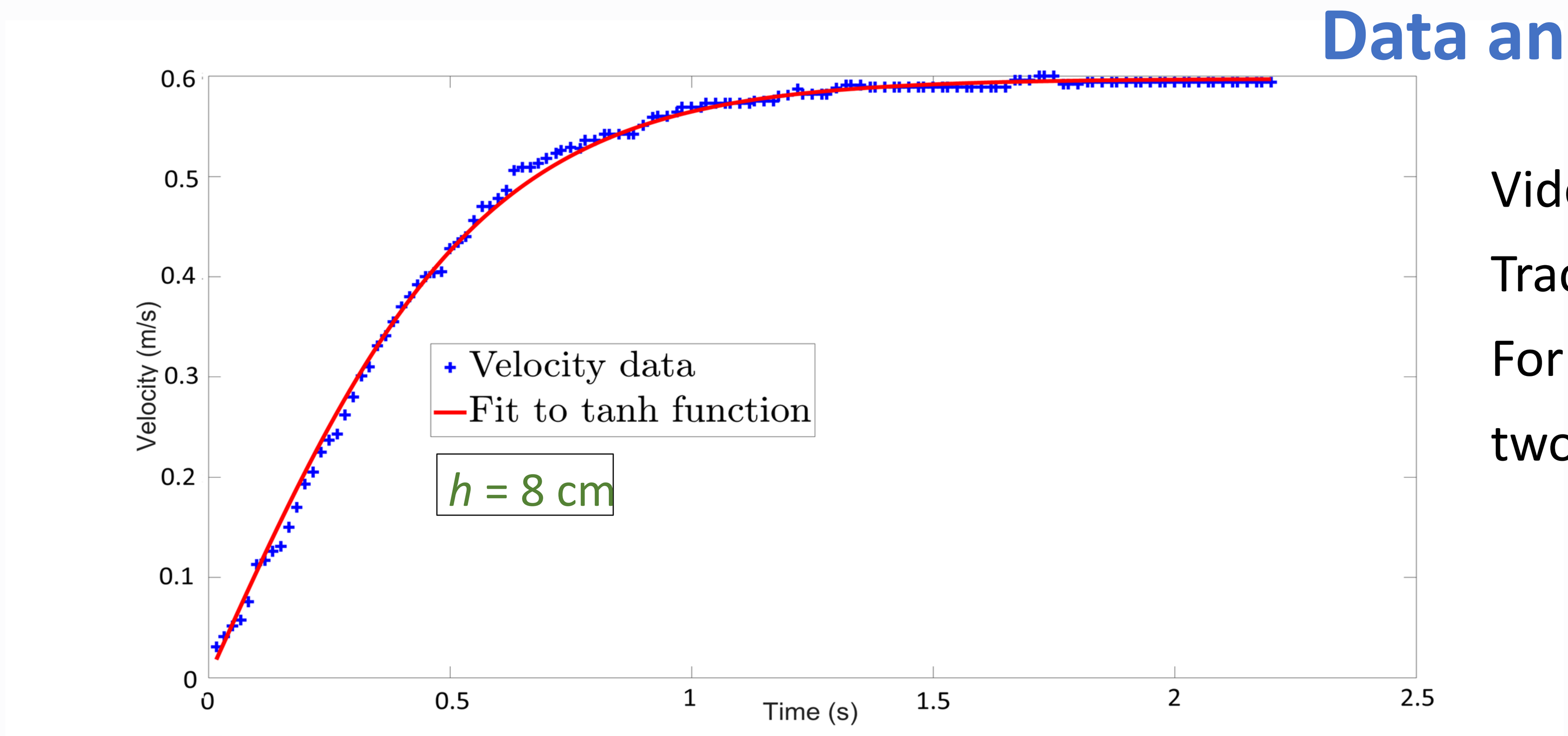


Fig. 1

Videos of transient flow with different levels of water were analyzed with Tracker (open source).

For each value of h, $v_2(t)=v_s\,\tanh t/\tau$ was fitted to the data with two free parameters: v_s and τ (see Fig. 1).

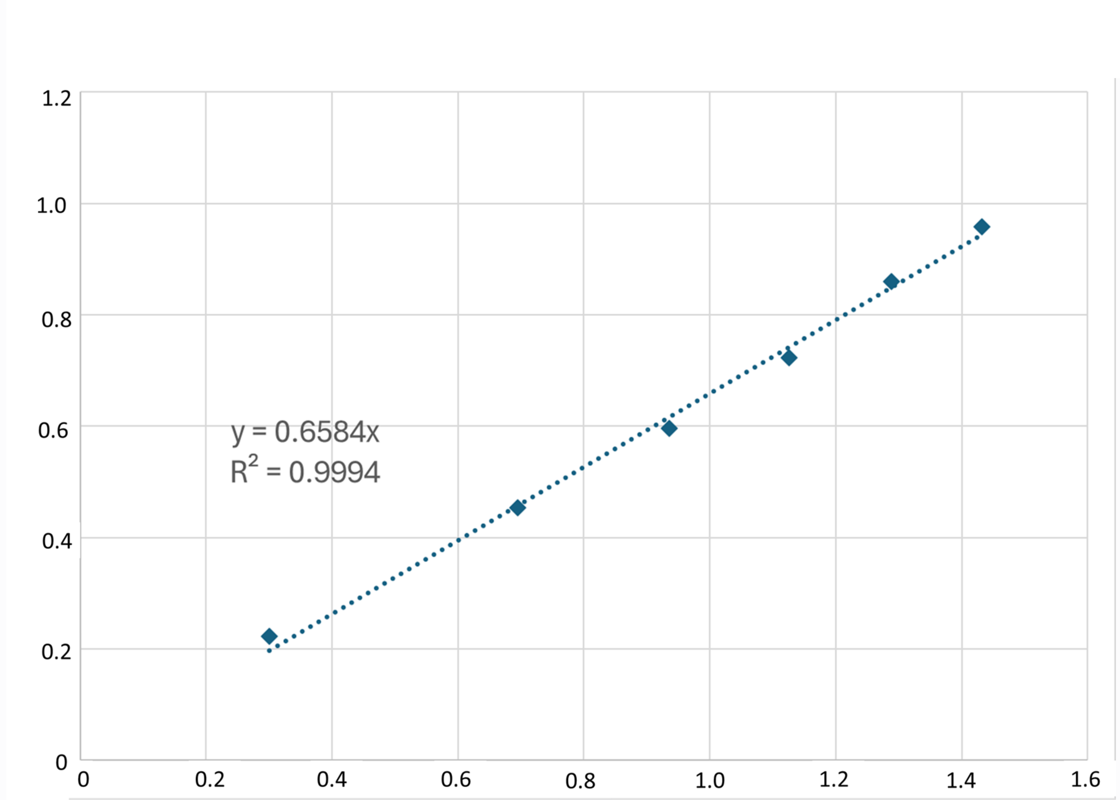


Fig. 2: $v_s(\mathrm{m/s})$ versus $\sqrt{2g(h-h_0)}$ with $h_0 = 0.035$ m

The regime velocity does not agree with Torricelli’s law. Instead, it obeys the law (see Fig. 2):

$$v_s=\sqrt{2g(h-h_0)/(1+\gamma)}$$

that follows from the steady Bernoulli’s equation with losses in the form:

$$gh=\frac{1}{2}\,(1+\gamma)v_s^2+gh_0.$$

Parameters γ and h_0 correspond to energy loss proportional to v_s^2, and independent of velocity, respectively. From our analysis : $h_0 \approx 35$ mm and $\gamma \approx 1.3$. It can be shown that the functional form of the transient velocity is not affected by the presence of losses. The corresponding formula is:

$$v_2(t)=\sqrt{2g(h-h_0)/(1+\gamma)}\,\tanh\left(\sqrt{\frac{g(h-h_0)(1+\gamma)}{2}}\frac{t}{L}\right)$$

Conclusions

This study demonstrates a low-cost experimental verification of the unsteady Bernoulli’s equation using a smartphone camera and Tracker software. The results confirm that the transient flow velocity from a pipe connected to a tank follows the predicted hyperbolic tangent functional form. While the nonviscous model accurately represents the transient flow, incorporating energy losses is essential to reconcile experimental parameters and steady-state velocities with theoretical values. From an educational perspective, the experiment is highly effective as a Predict-Observe-Explain (POE) activity. Students typically will predict a step-like transition to steady state; observing the actual transient phase prompts a deeper investigation into the mathematical modeling of unsteady flows. For advanced students, the project serves as a practical application of complex fluid dynamics, highlighting the limitations of the steady-state assumptions found in introductory courses. Ultimately, the apparatus provides a transparent and accessible bridge between idealized theory and real-world hydrodynamic behavior.

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