CHASING NON-DECOMPOSABLE POSITIVE MAPS WITH SDPs

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INTRODUCTION

Positive (but not completely positive maps) lie at the heart of entanglement detection [1].
They extend the Peres-Horodecki criterion, providing a broader way to test whether a quantum state is entangled beyond partial transposition.

Non-decomposable maps are special: they can detect Positive Partial Transpose (PPT) entangled states, which remain hidden from decomposable witnesses. Since Choi pioneering example [2], few explicit constructions exist, and a general framework is still missing [3].

We introduce a systematic method to generate new non-decomposable positive maps using semidefinite programming (SDP), providing a framework to explore a larger set of entanglement witnesses.

METHOD

Separability is difficult to impose in an optimization procedure.

EXTENDIBILITY

We can approximate the separability condition [4] imposing that the state ρ_{AB} can be extended by a symmetric (and PPT) state $\rho_{AB_1\ldots B_k}$:

$$\rho_{AB} = \mathrm{Tr}_{B_2\ldots B_k}(\rho_{AB_1\ldots B_k})$$

$$\rho_{AB_1\ldots B_k} = \rho_{AB_{\pi(1)}\ldots B_{\pi(k)}} \quad \forall \pi \in \mathbb{P}_k$$ — Permutations of k elements.

$$\rho_{AB_1\ldots B_k}^{T_l} \succeq 0 \quad \forall l$$ — Positivity of partial transpose on $B_1, \ldots, B_l$

ρ_{AB} separable $\Longrightarrow$ ρ_{AB} k-symmetric extendible for finite k.

POSITIVITY

$$\zeta_k = \min_{\rho_{AB}\in S_k} \mathrm{Tr}(\rho_{AB} C_\Phi) \geq 0$$

S_k: Set of k-symmetric extendible states

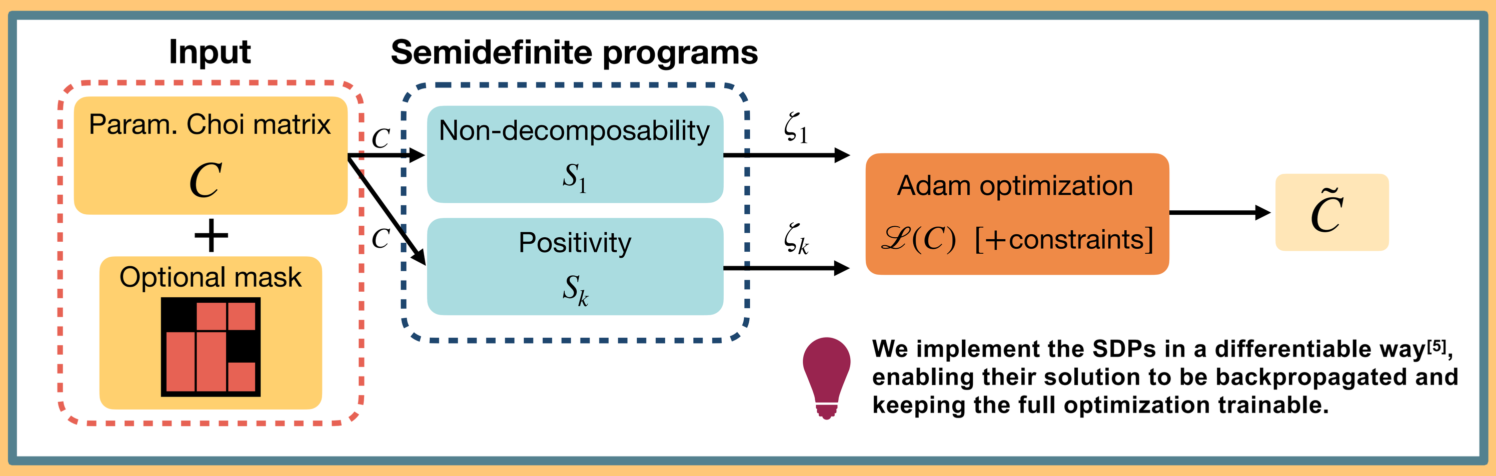
Prod $\subseteq$ Sep $\subseteq$ S_k → If $\zeta_k > 0$, then Φ is positive

NON-DECOMPOSABILITY ($k = 1$)

If $\zeta_1 < 0$ it means that C_Φ is negative on some PPT entangled states, so it cannot be decomposable.

These conditions can be expressed as a semidefinite program (SDP) a convex optimization problem that can be computed efficiently.

OPTIMIZATION PROCESS



Loss function : $L(C_\Phi) = \mathrm{ReLU}(\epsilon + \zeta_1(C_\Phi)) + \gamma \mathrm{ReLU}(-\zeta_k(C_\Phi))$

THEORY

CHOI-JAMIOŁKOWSKI ISOMORPHISM

Every linear map Φ has a unique state representation:

MAP	CHOI STATE
$\Phi : B(\mathscr{H}) \to B(\mathscr{H})$	$C_\Phi = \frac{1}{d}\sum_{ij} \Phi(\lvert i\rangle\langle j\rvert) \otimes \lvert i\rangle\langle j\rvert$

POSITIVITY

A map is positive if it maps PSD (positive semidefinite) operators to PSD operators:.

$$\rho \geq 0 \implies \Phi(\rho) \geq 0 \qquad \mathrm{Tr}(C_\Phi \rho_A \otimes \rho_B) \geq 0$$

NON-DECOMPOSABILITY

A map Φ is non-decomposable if it cannot be written as the sum of a completely positive map and a completely copositive map:

$$\Phi \neq \Phi_{CP} + \Phi_{CoP}$$

- **$\Phi_{CoP} \circ T$ is CP**
- **$\Phi_{CP} \otimes \mathbb{I}_n$ is P for all $n \geq 1$**

$$\exists \rho_{AB} \text{ PPT s.t. } \mathrm{Tr}(C_\Phi \rho_{AB}) < 0$$

Positive under partial transposition $\rho_{AB}^{T_B} \geq 0$

In d ≤ 2, all positive maps are decomposable.

APPLICATIONS

SOLVING OPEN CONJECTURES

Many questions in this field are open due to the difficulty of finding some elusive counterexamples in the space of non-decomposable maps.

Two examples of questions that we settled using this method:

→ **This bound [6], where $\min \mathrm{Re}[\sigma(\Phi)]$ is the smallest eigenvalue of Φ, holds for any 2-positive map.**

$$\mathrm{Tr}\,\Phi \leq d \min \mathrm{Re}[\sigma(\Phi)] + d^2 - d,$$

Is there any non-decomposable map in d=3 that violate it? YES!

→ **Are there non-decomposable maps $\Phi : \mathcal{B}(\mathbb{C}^2) \to \mathcal{B}(\mathbb{C}^m)$ with $m \geq 4$? [7]** YES! (PLENTY!)

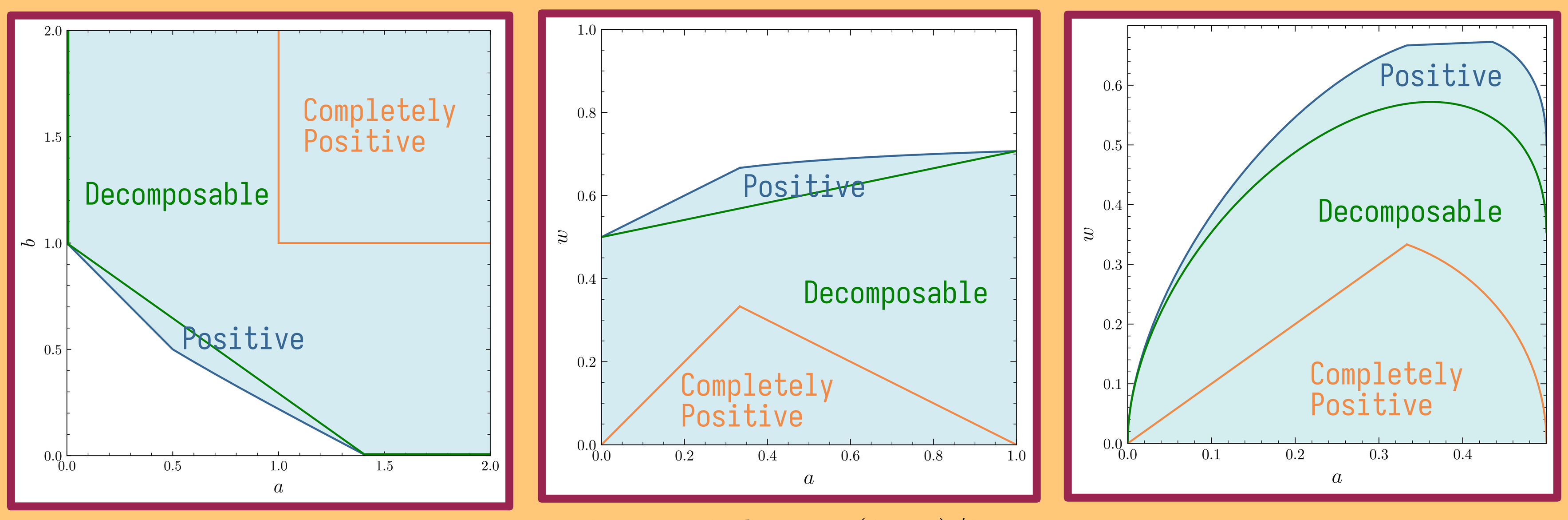
FINDING NEW FAMILIES OF NON-DECOMPOSABLE MAPS

Only relatively few parametrized classes of non-decomposable maps are known in the literature.
This tool can point us to new classes satisfying specific properties.

→ **Example of a new class of non-decomposable maps in d=3.**

$$\Phi(\rho) = \begin{pmatrix} a\rho_{11} + c\rho_{22} + b\rho_{33} & w^*\rho_{21} & z\rho_{13} \\ w\rho_{12} & b\rho_{11} + a\rho_{22} + c\rho_{33} & 0 \\ z^*\rho_{31} & 0 & c\rho_{11} + b\rho_{22} + a\rho_{33} \end{pmatrix}$$

The numeric optimization suggests the right form.
The family can then characterized analitically.



$b = c \geq 0$ and $|w| = |z| = 1$ | $0 \leq a \leq 1$, $b = c = (1-a)/2$ and $w = z \geq 0$ | $0 \leq a = c \leq 1/2$, $b = 1 - 2a$ and $w = z \geq 0$

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