Integrability of the Thirring Quantum Cellular Automaton

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Supervisors: Paolo Perinotti
Alessandro Bisio

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16/09/2026

UNIVERSITÀ
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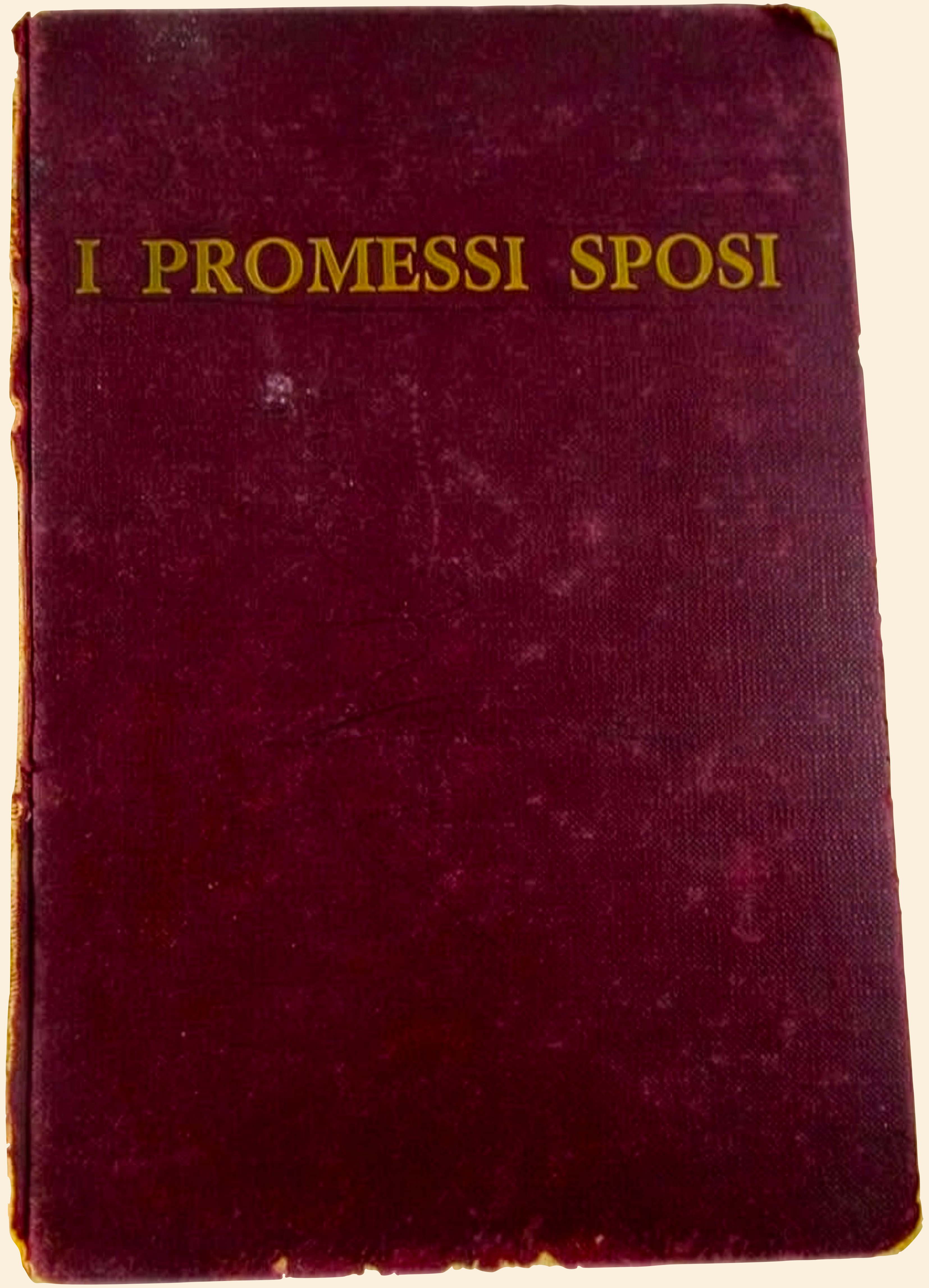
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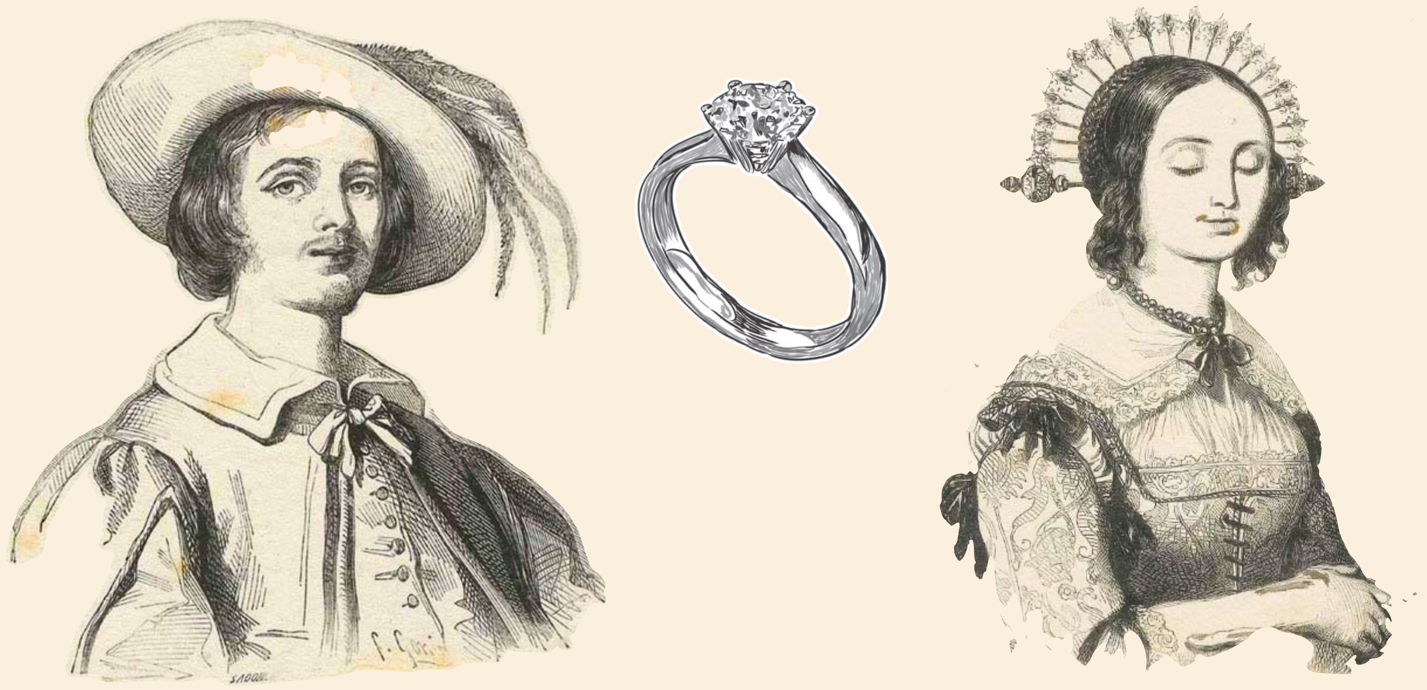
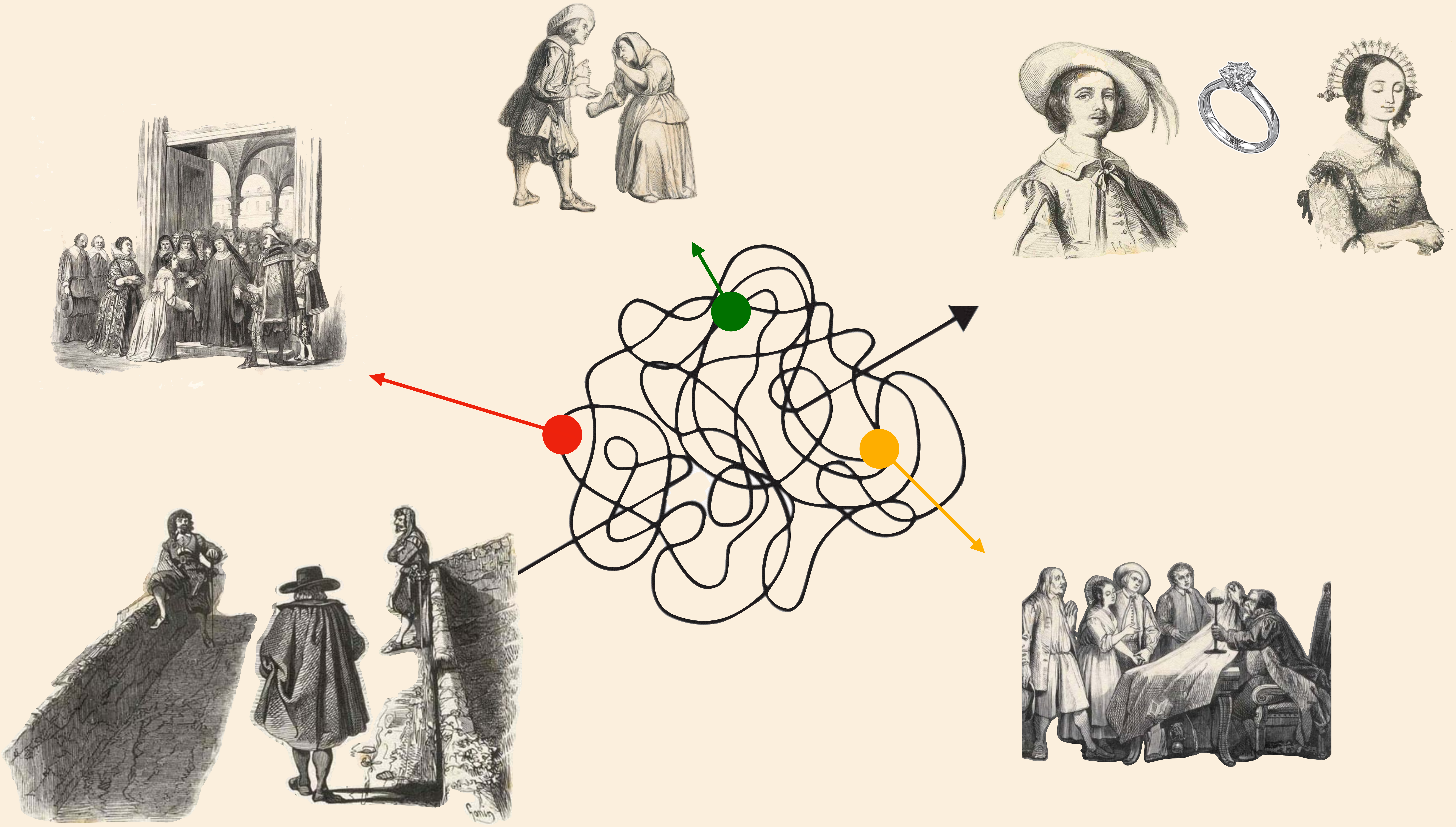
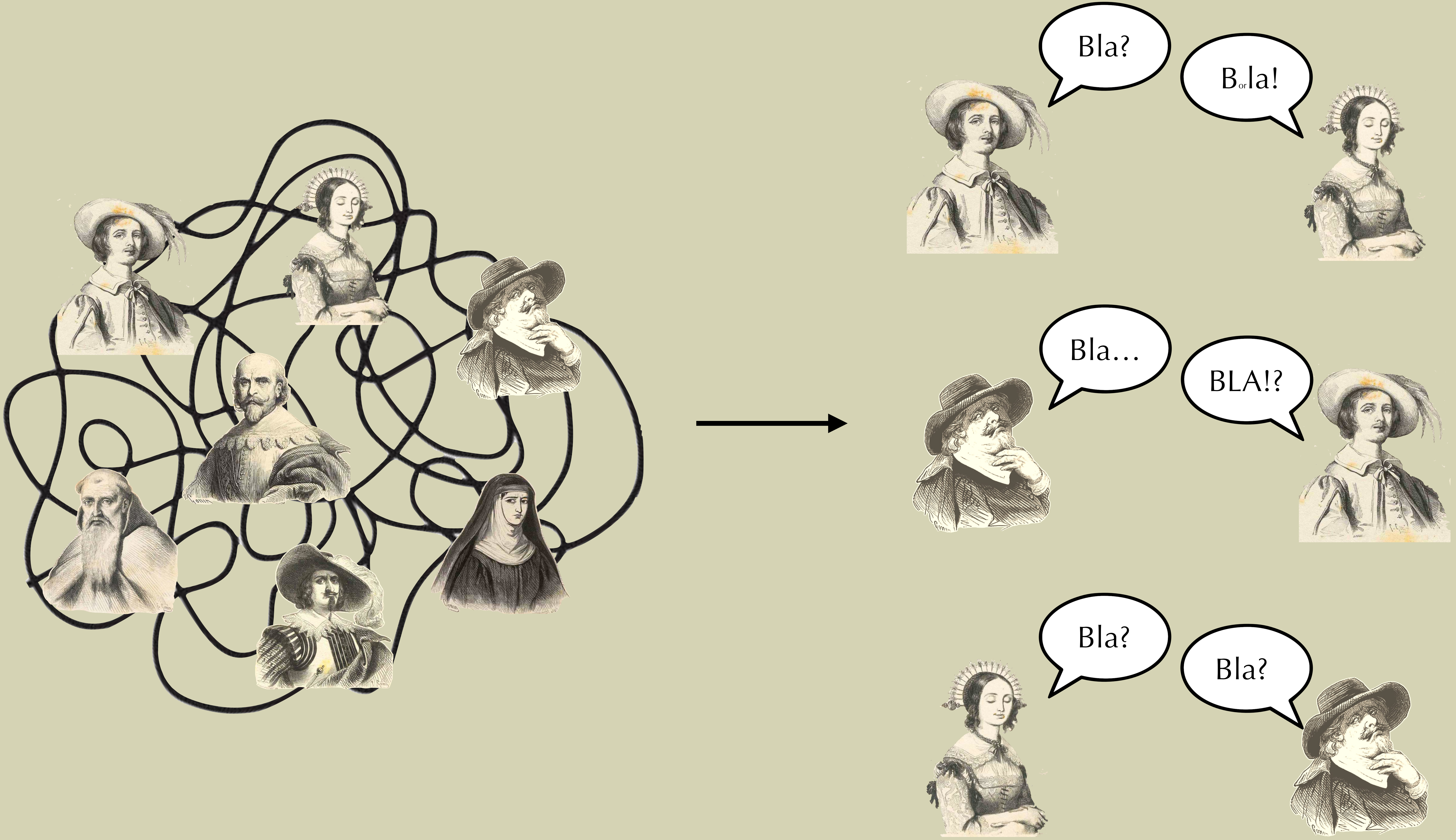
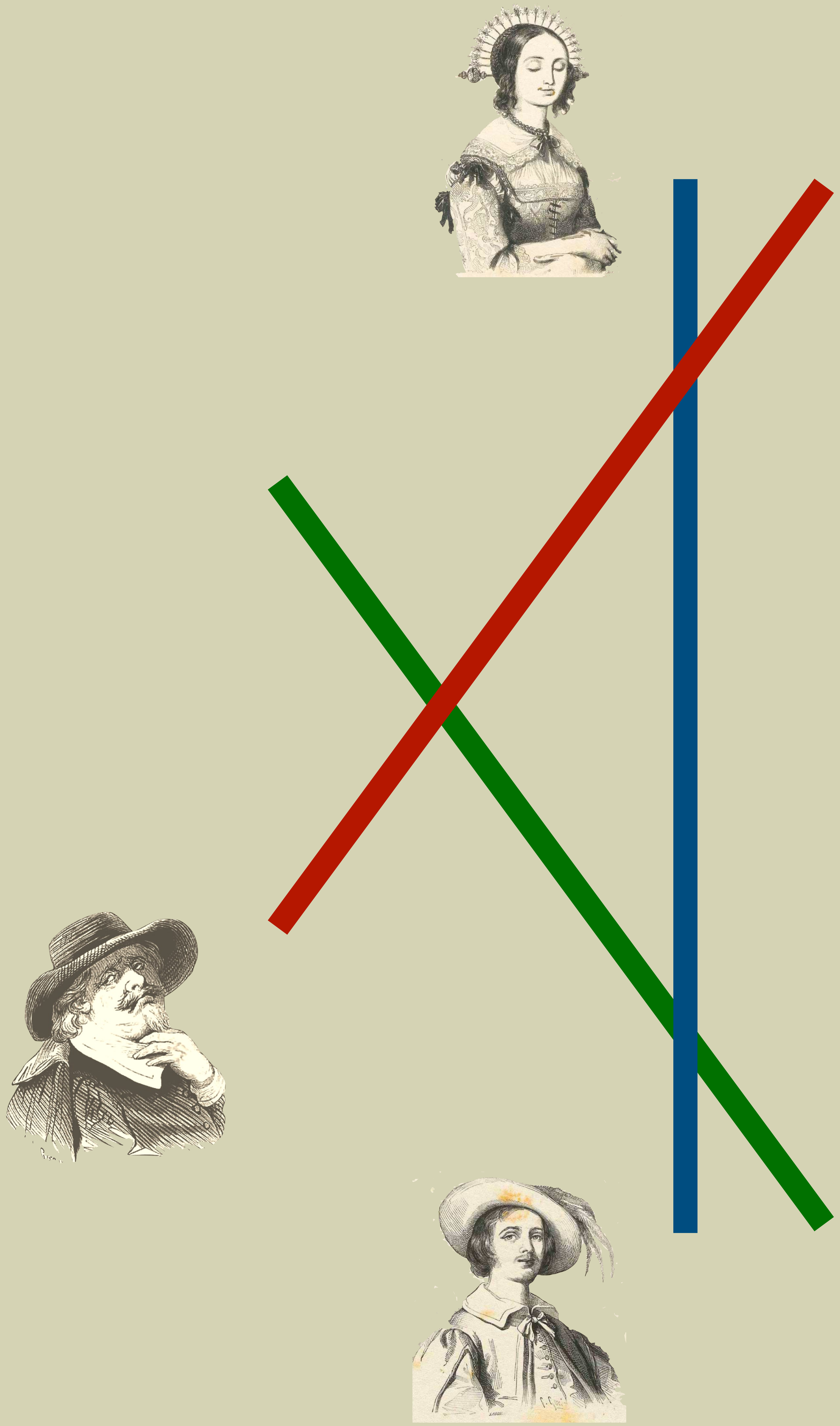





Bla?
Borla!
Bla…
BLA!?
Bla?
Bla?

if



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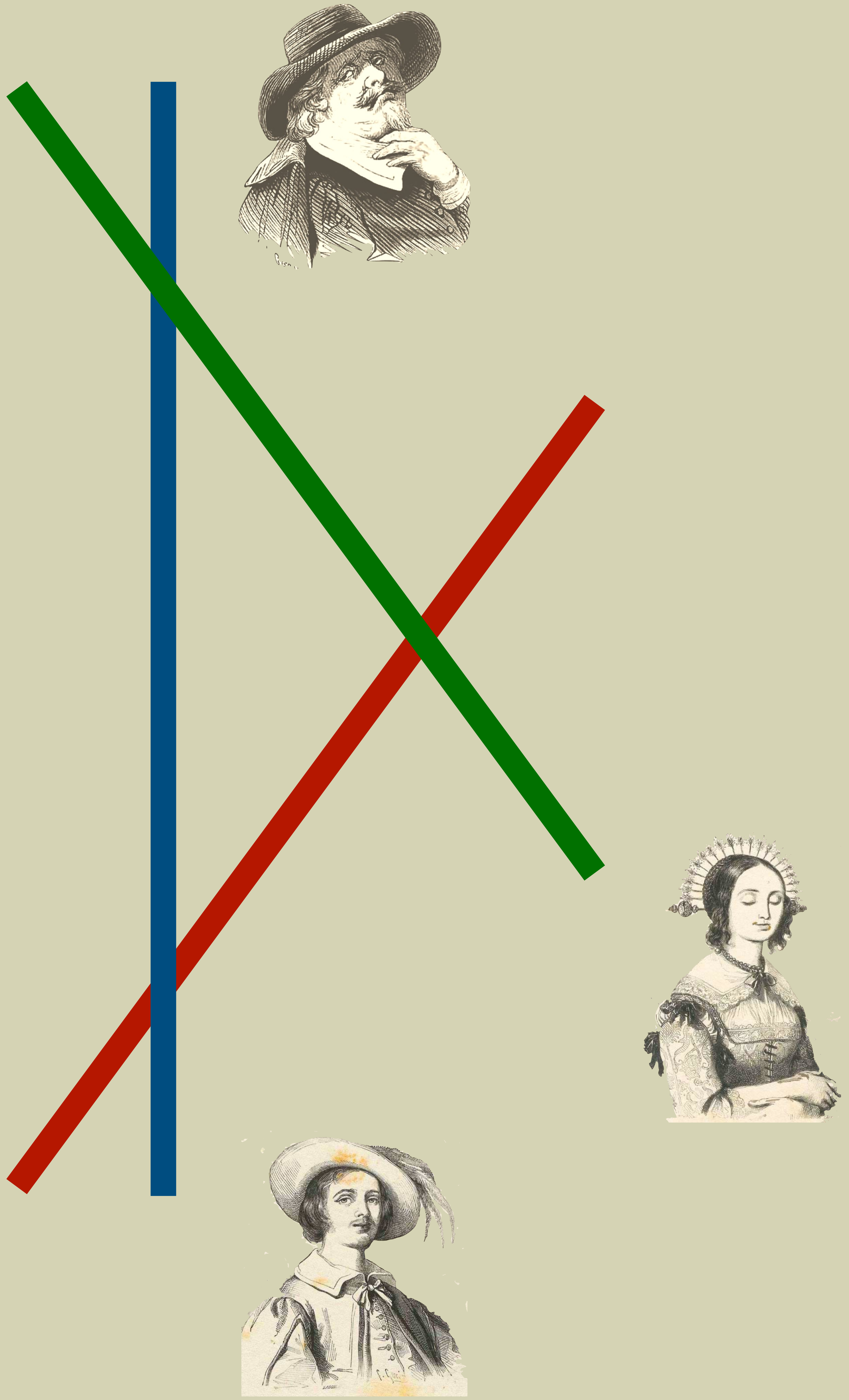


$\mapsto$

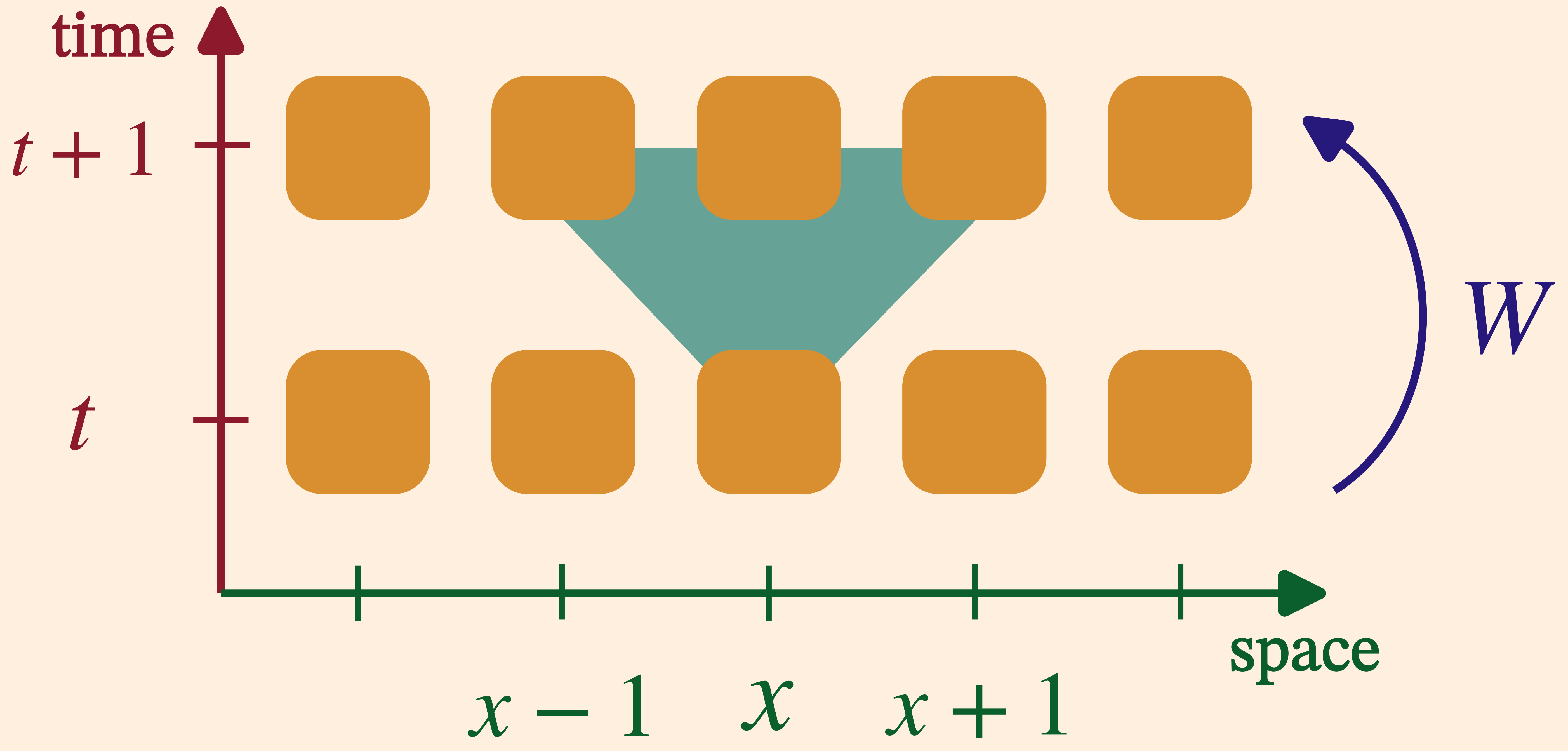
Quantum
Cellular
Automaton

Quantum **C**ellular **A**utomaton

most general:

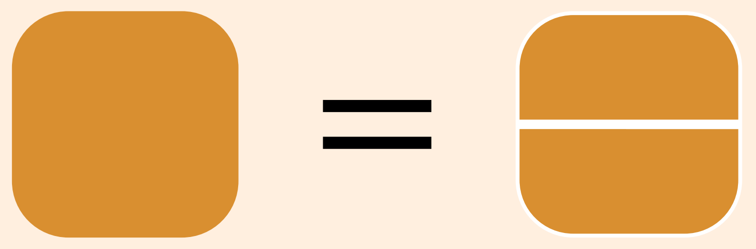
- **Unitary**
- **Local**
- **Time-discrete**

dynamics of a **lattice** of **quantum systems**



Dirac QCA

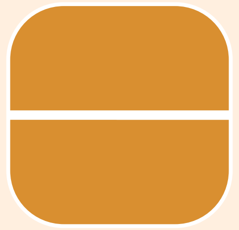
Cell



$\simeq \mathbb{C}^2$

Dirac QCA

Cell

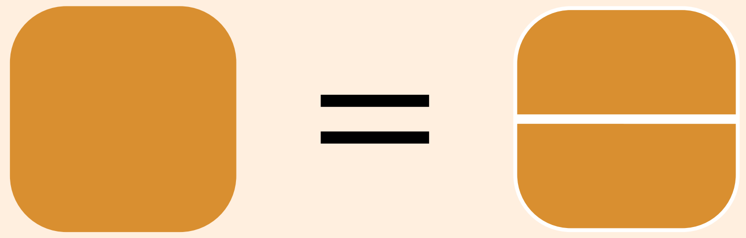
 $=$  $\simeq \mathbb{C}^2$

Update

$$W = \begin{bmatrix} nT & im\square \\ im\square & nT^\dagger \end{bmatrix}$$

Dirac QCA

Cell

 $\simeq \mathbb{C}^2$

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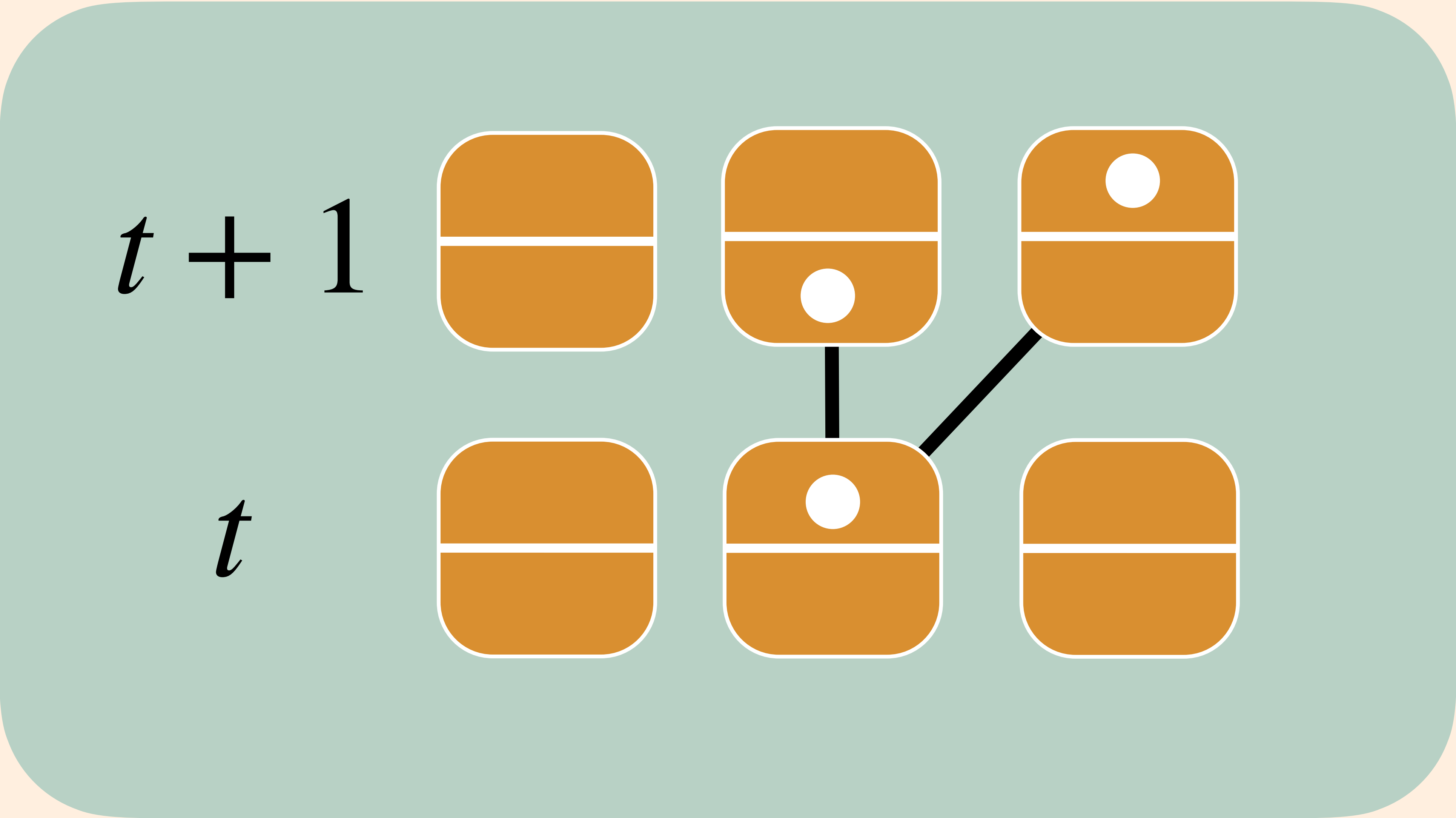
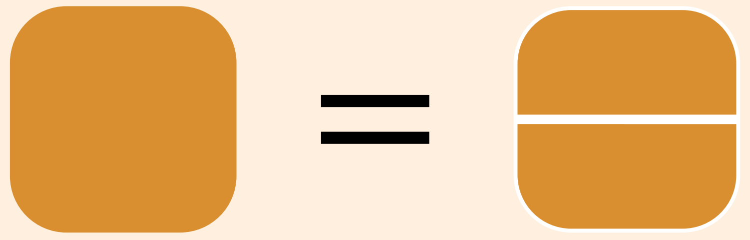


Dirac QCA

Cell

 $\simeq \mathbb{C}^2$

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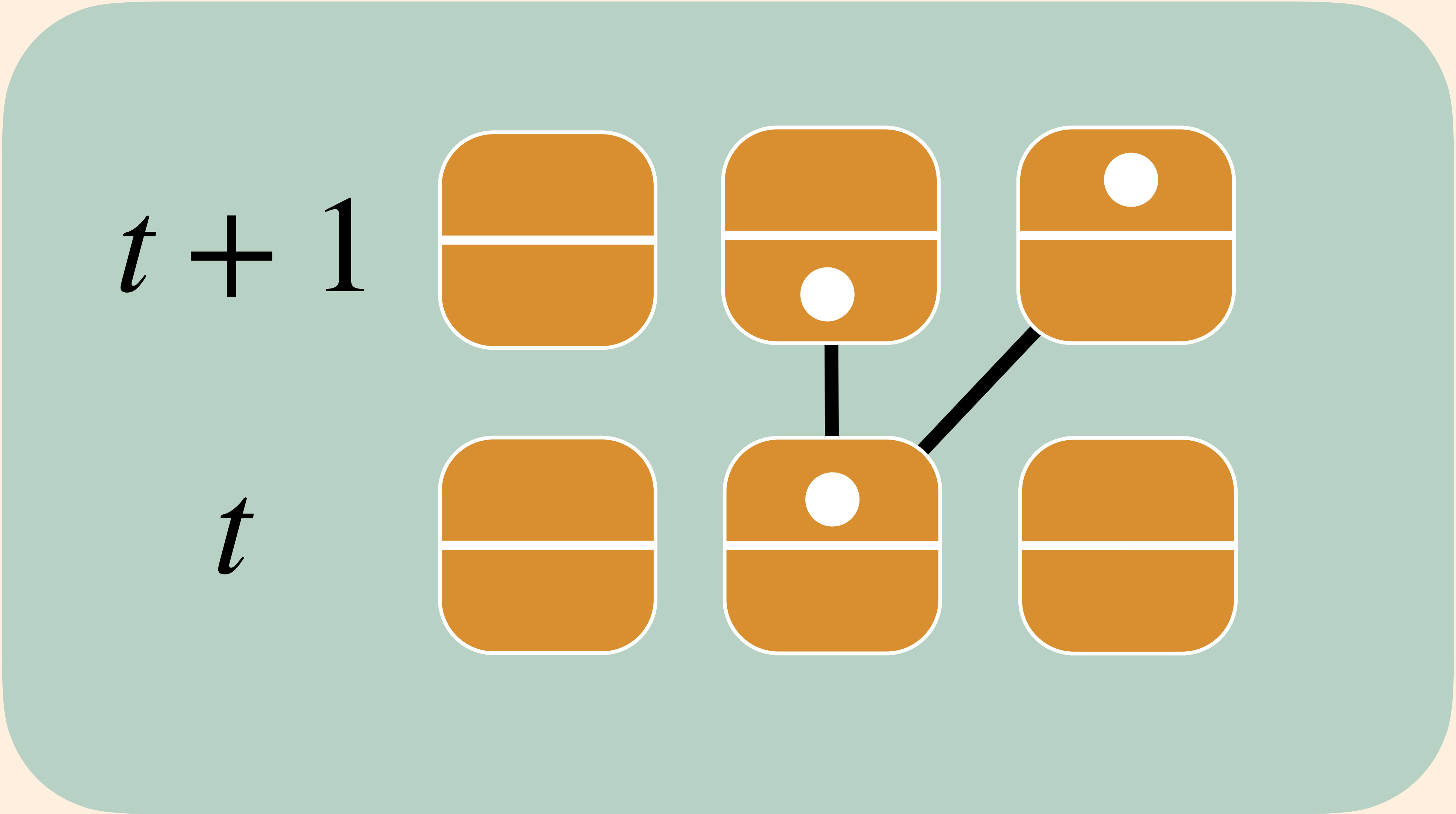
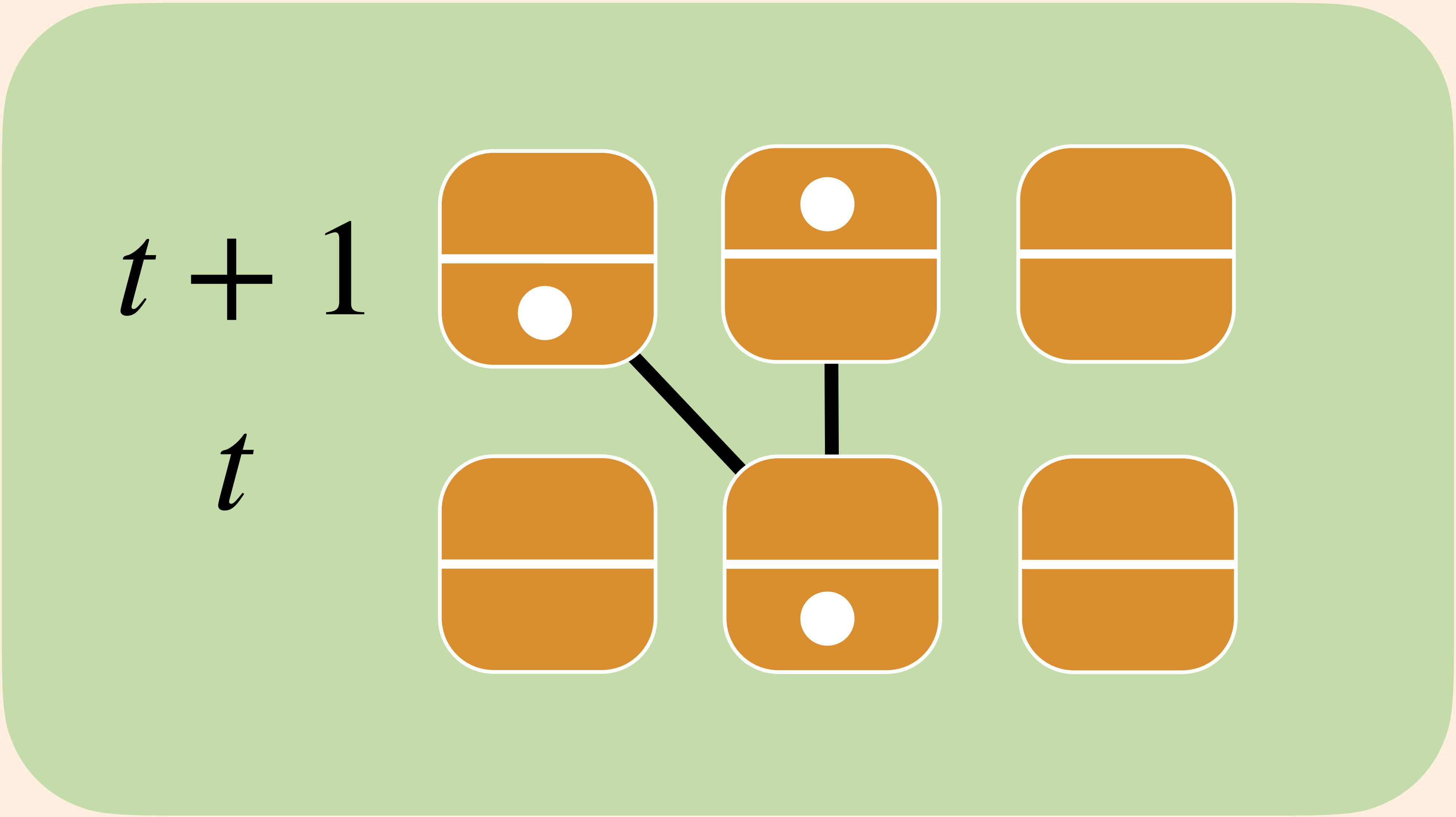
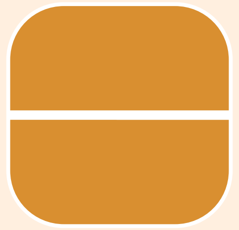




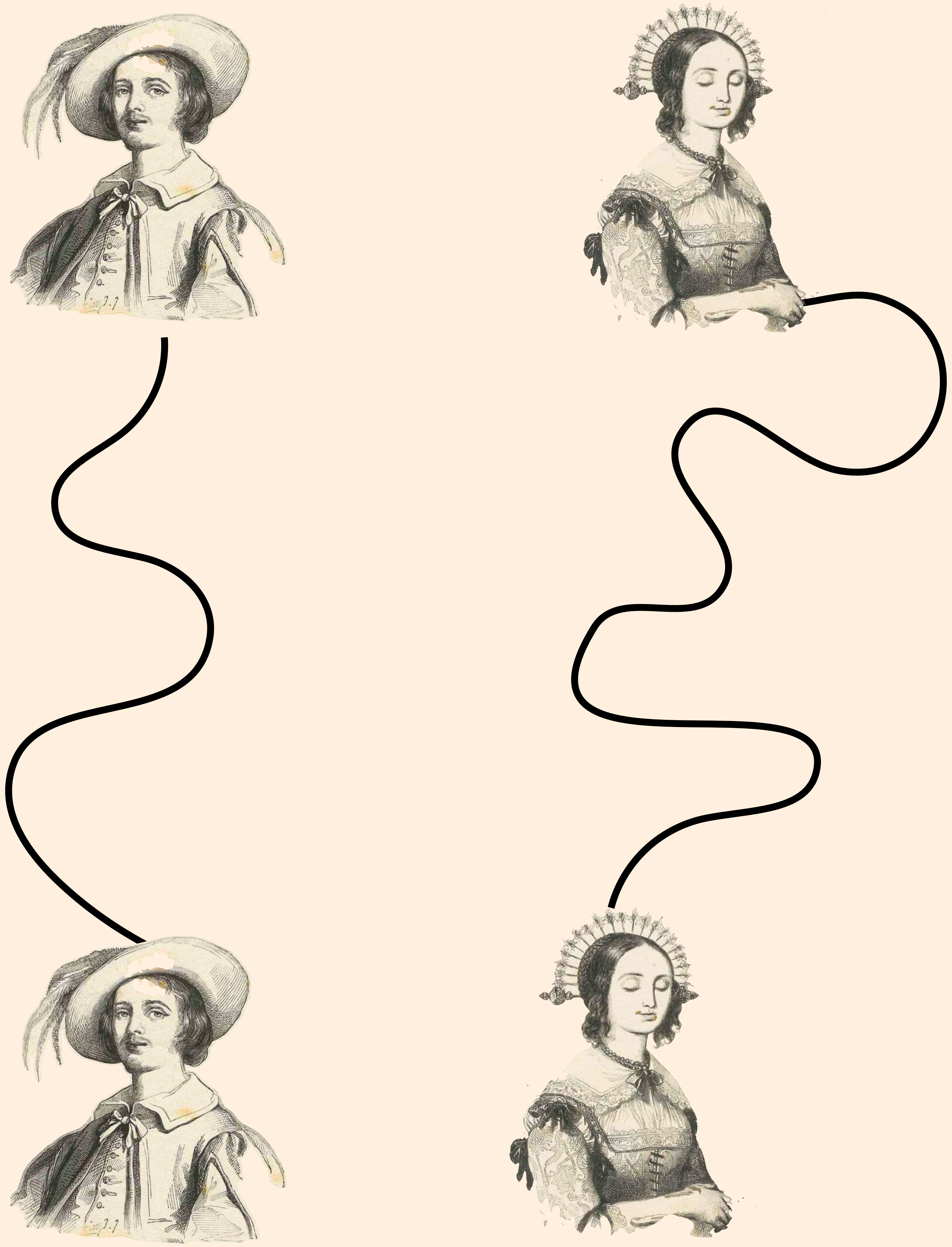
Dirac QCA

Cell

 $=$  $\simeq \mathbb{C}^2$

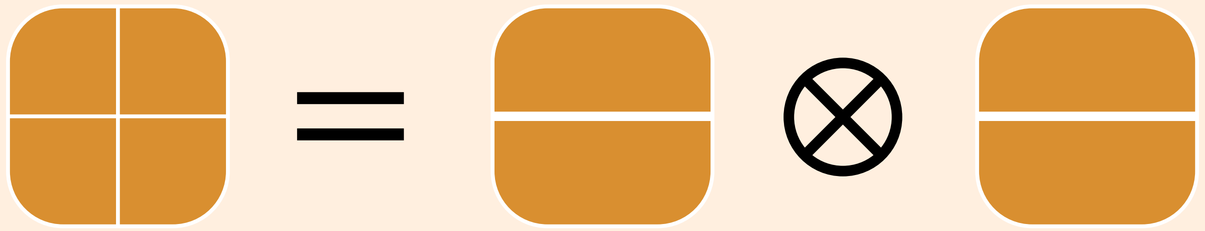
Update

$$W = \begin{bmatrix} nT & im\square \\ im\square & nT^\dagger \end{bmatrix}$$



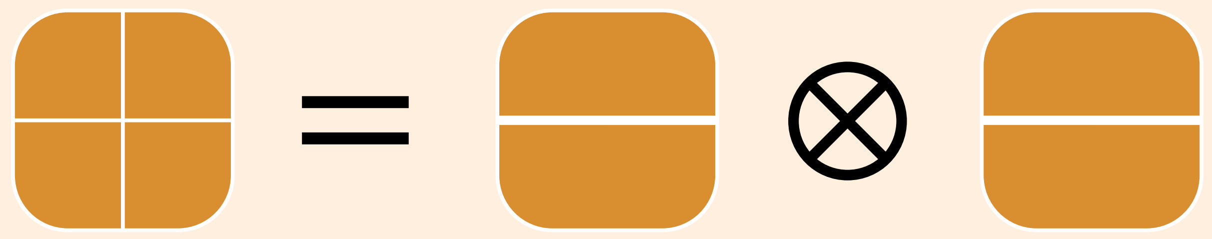
Thirring QCA

Cell



Thirring QCA

Cell

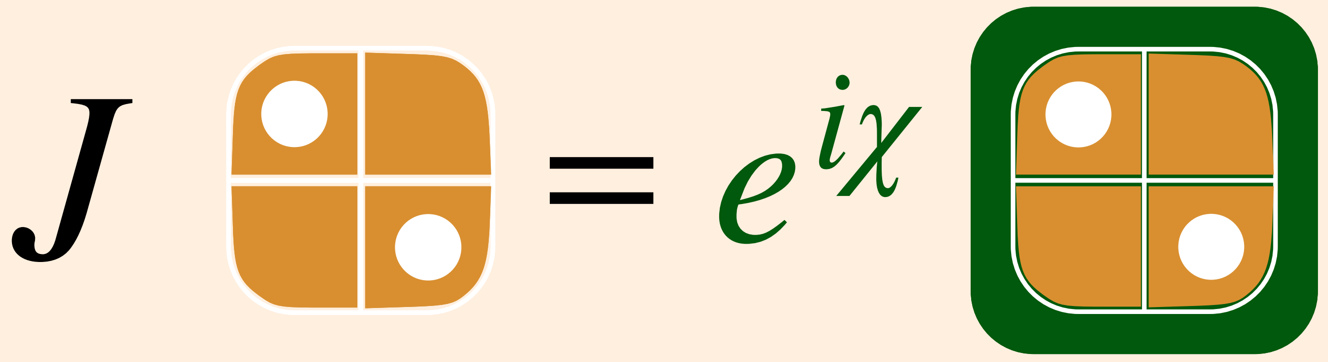


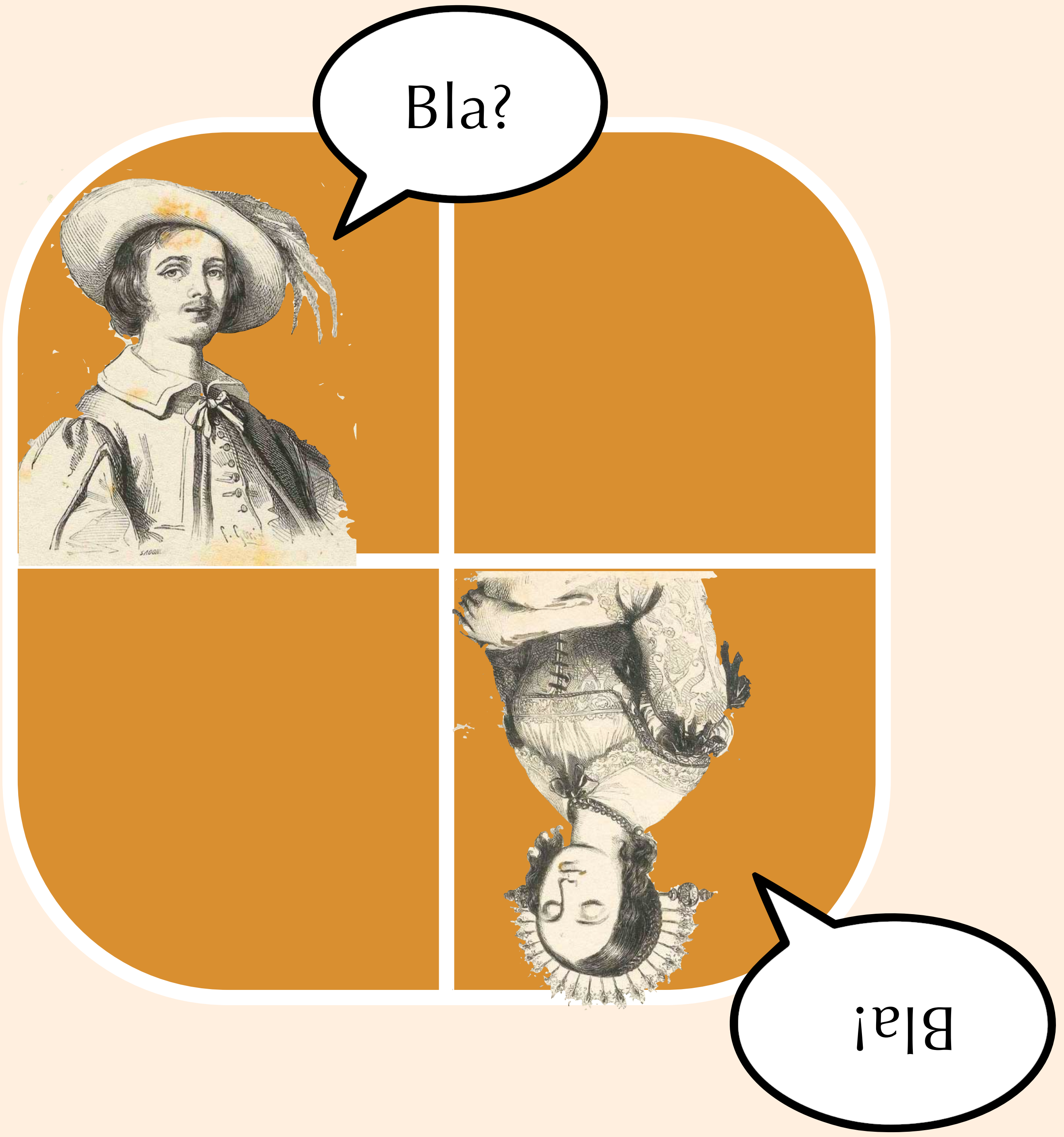
Update

$$U_2 = J(\chi)W^{\otimes 2}$$

Same site & opposite spins

= **interaction**





Can many-particle scattering be built *consistently* from two-particle collisions?

Solve the interacting eigenvalue equation:

$$J(\chi)W^{\otimes N}|\Psi\rangle = \Lambda|\Psi\rangle$$

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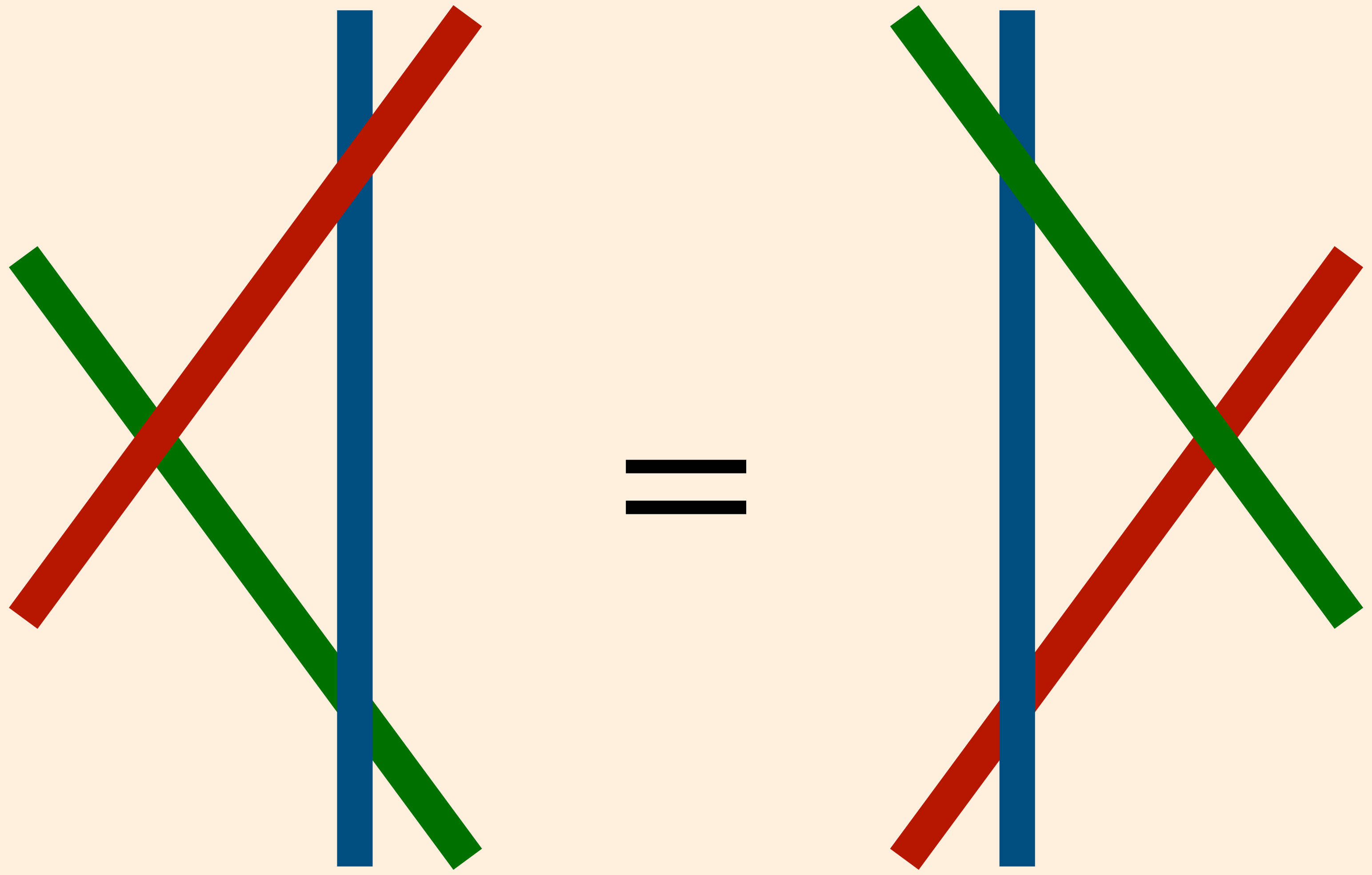
Encode coefficient relations into an R matrix…

Solve the interacting eigenvalue equation:

$$J(\chi)W^{\otimes N}|\Psi\rangle = \Lambda|\Psi\rangle$$

Encode coefficient relations into an R matrix…

… check consistency of collision sequences



$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

Coordinate
Bethe
Ansatz

Coordinate Bethe Ansatz

Starting point:

Superposition of free eigenstates $|\Phi_j\rangle$ with the same eigenvalue Λ.

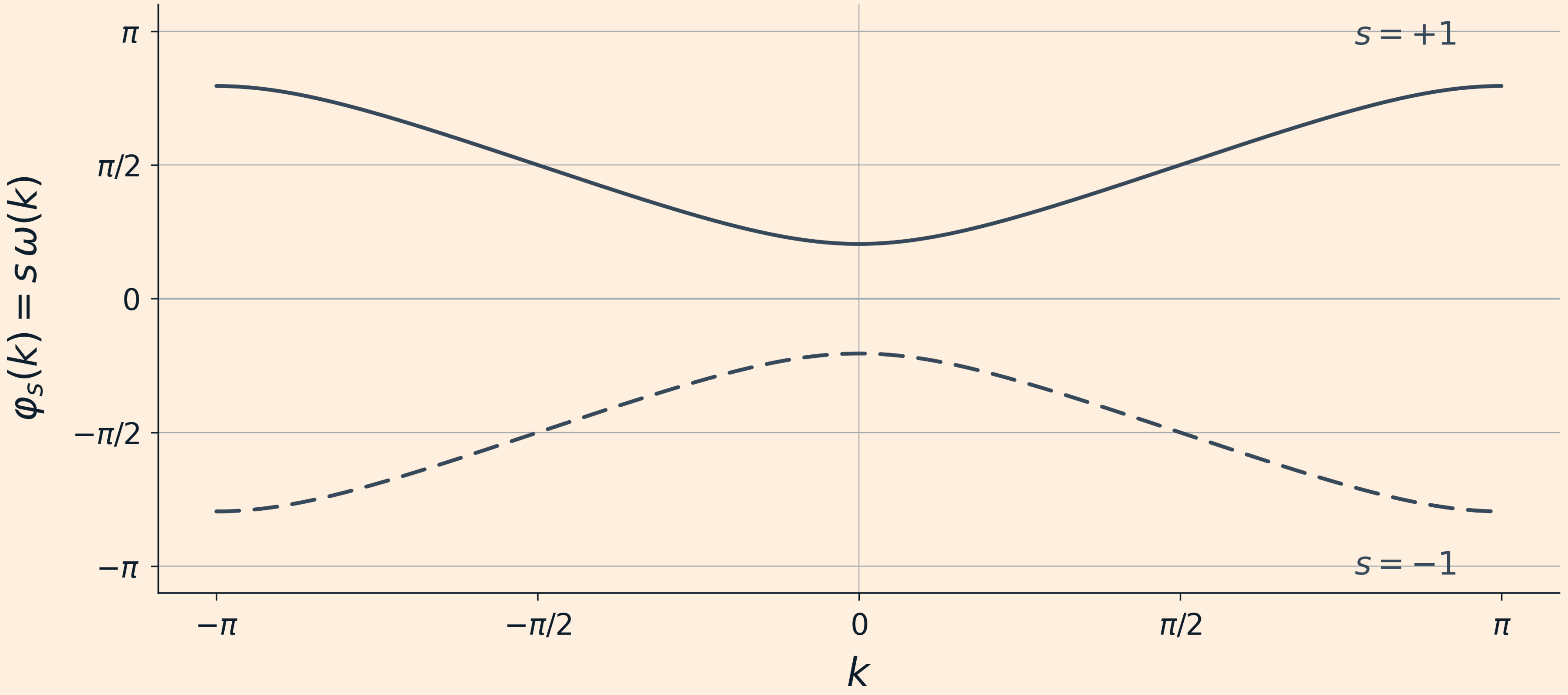
$$W^{\otimes N} | \Phi_j (\mathbf{k}, \mathbf{s})\rangle = \Lambda(\mathbf{k}, \mathbf{s}) | \Phi_j (\mathbf{k}, \mathbf{s})\rangle$$

Scattering can mix these states.

One-particle spectrum

$$\lambda(s,k) = e^{is\omega(k)}$$

$$\omega(k) = \arccos[n\cos k]$$

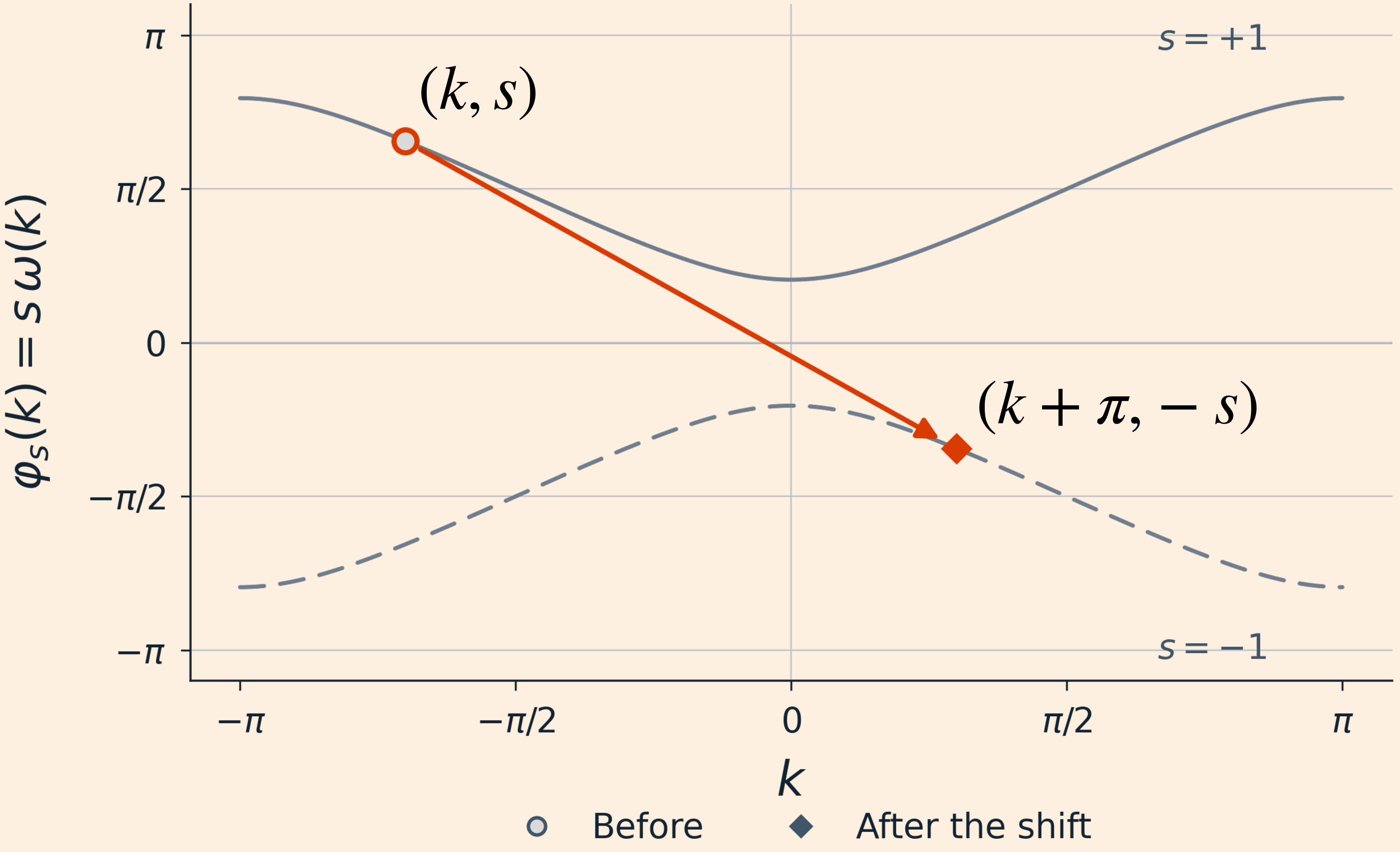


One-particle spectrum

○ $= (k, s)$

◇ $= (k + \pi, -s)$

$$\lambda(\bigcirc) = -\lambda(\Diamond)$$



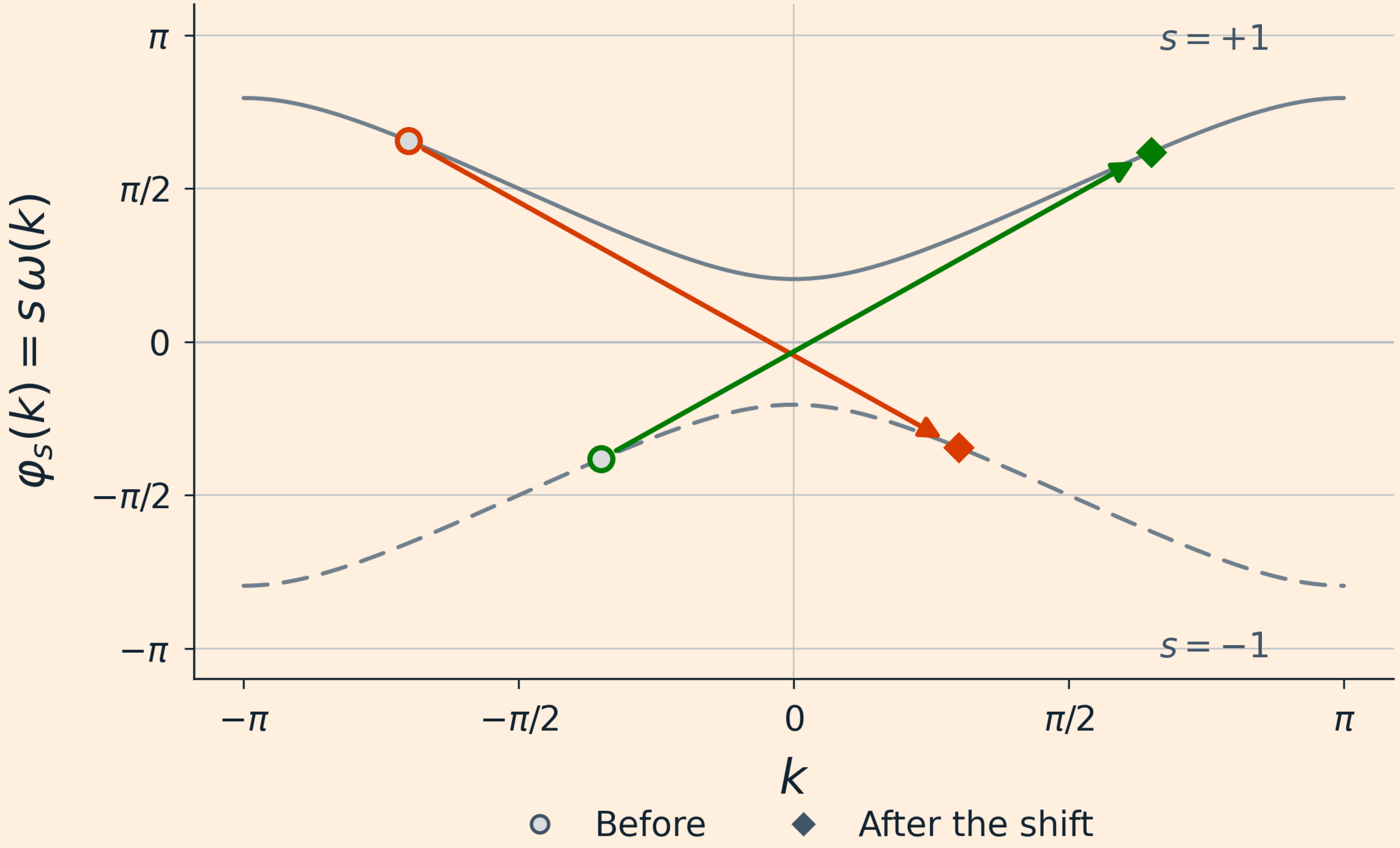
Two-particle degeneracy

$$\Lambda = \lambda_1 \lambda_2$$

$$\lambda(\bigcirc) = -\,\lambda(\Diamond)$$

$$\Downarrow$$

$$\Lambda(\bigcirc, \bigcirc) = \Lambda(\Diamond, \Diamond)$$



Two-particle degeneracy

$$\Lambda = \lambda_1 \lambda_2$$

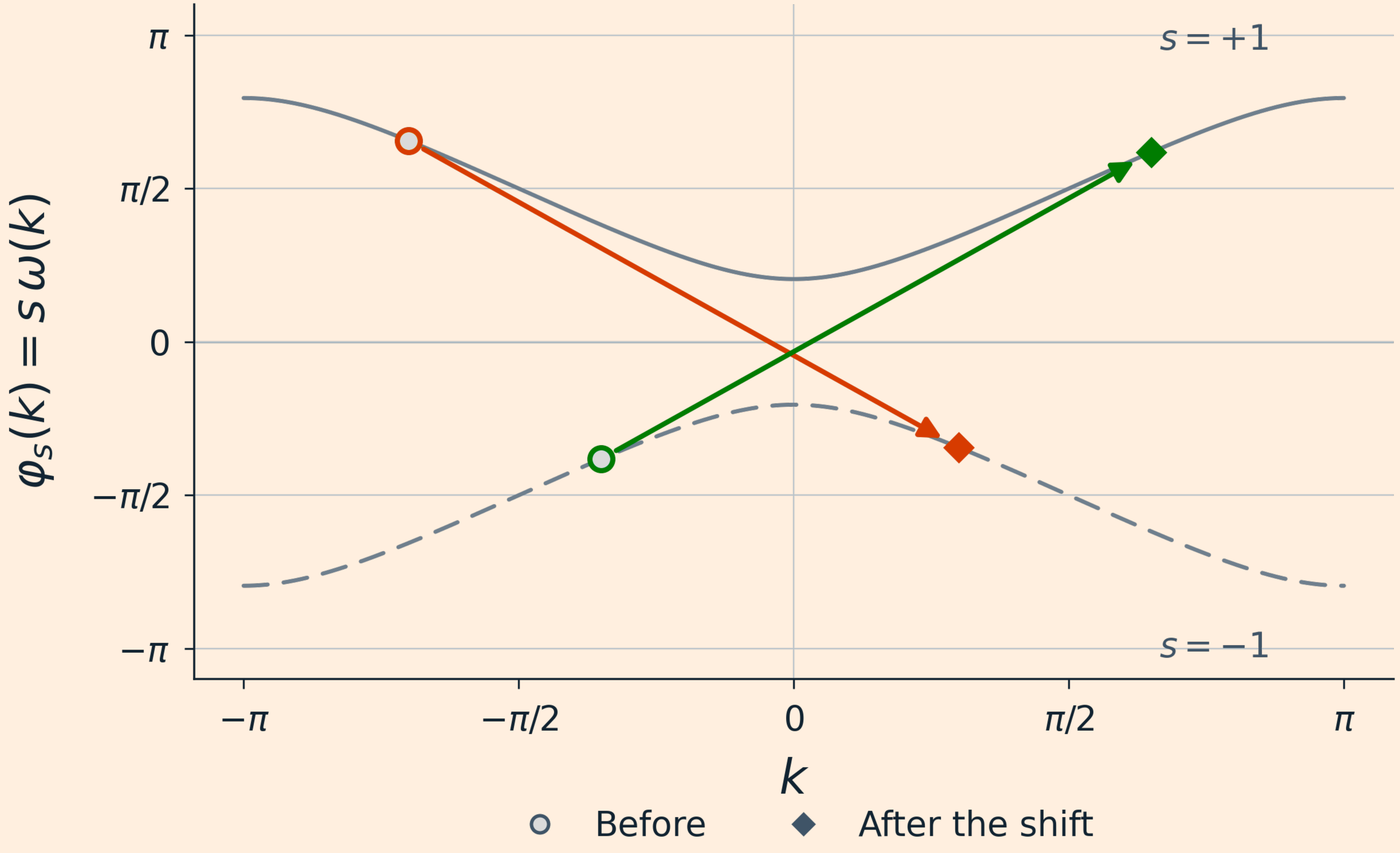
$$\lambda(\bigcirc) = -\lambda(\Diamond)$$

$$\Downarrow$$

$$\Lambda(\bigcirc,\bigcirc) = \Lambda(\Diamond,\Diamond)$$

$$\shortparallel \qquad\qquad\qquad \shortparallel$$

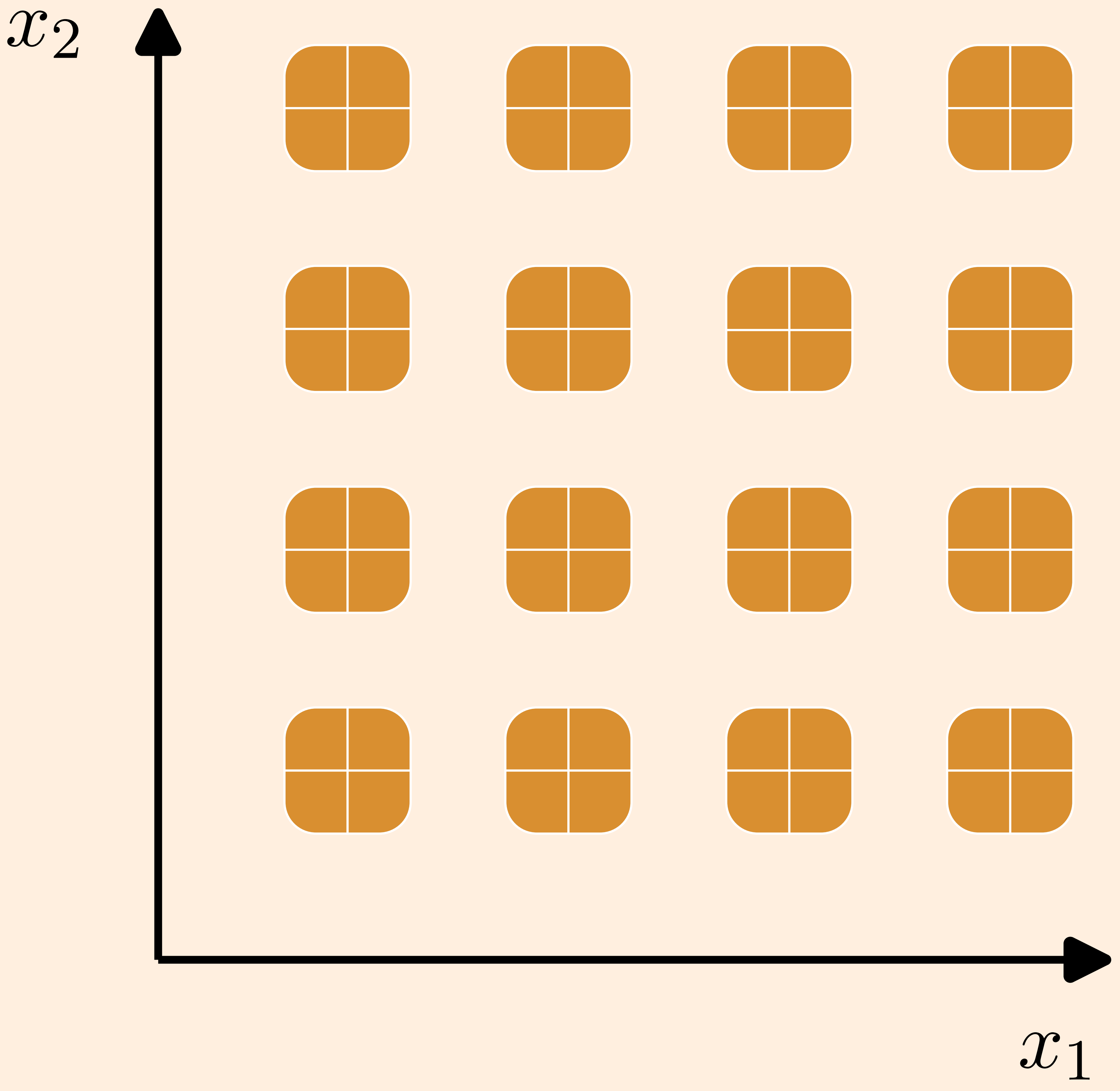
$$\Lambda(\bigcirc,\bigcirc) = \Lambda(\Diamond,\Diamond)$$

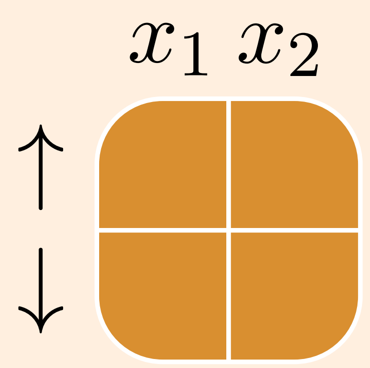


Two-particle degenerate state

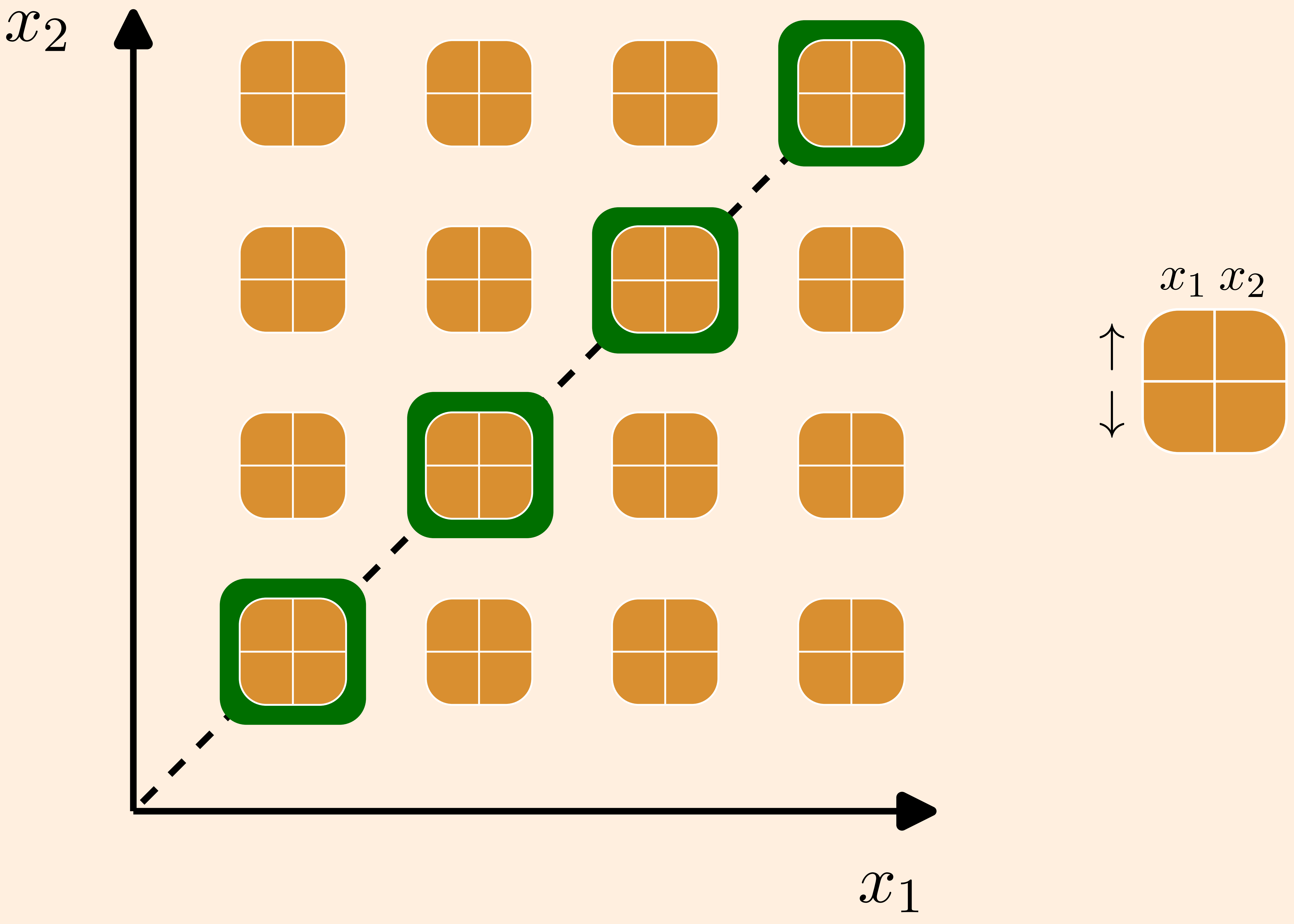
$$
\begin{aligned}
|\Phi(k_1, s_1; k_2, s_2)\rangle = \;& A_I^{00}\, |\Phi({\color{orange}\bigcirc}; {\color{green}\bigcirc})\rangle \\
+ \;& A_\Sigma^{00}\, |\Phi({\color{green}\bigcirc}; {\color{orange}\bigcirc})\rangle && \text{Exchange} \\
+ \;& A_I^{11}\, |\Phi({\color{orange}\Diamond}; {\color{green}\Diamond})\rangle && \text{Shift both} \\
+ \;& A_\Sigma^{11}\, |\Phi({\color{green}\Diamond}; {\color{orange}\Diamond})\rangle && \text{Exchange \& shift both}
\end{aligned}
$$

Coordinate Bethe Ansatz



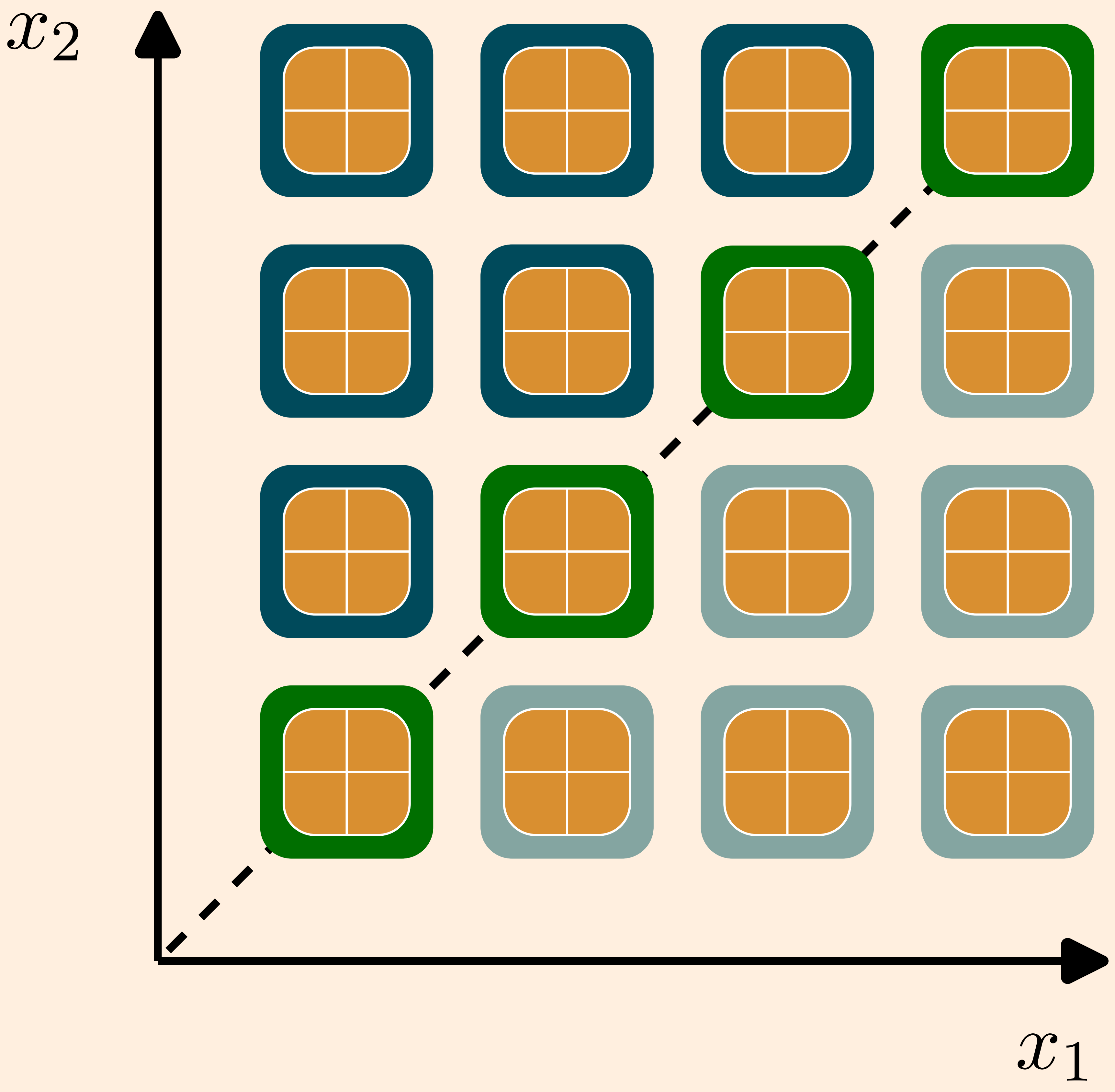


Coordinate Bethe Ansatz



$$|\Psi(\mathbf{k},\mathbf{s})\rangle = |\Psi\rangle_I + |\Psi\rangle_\Sigma + |\Psi\rangle_\partial$$

Coordinate Bethe Ansatz



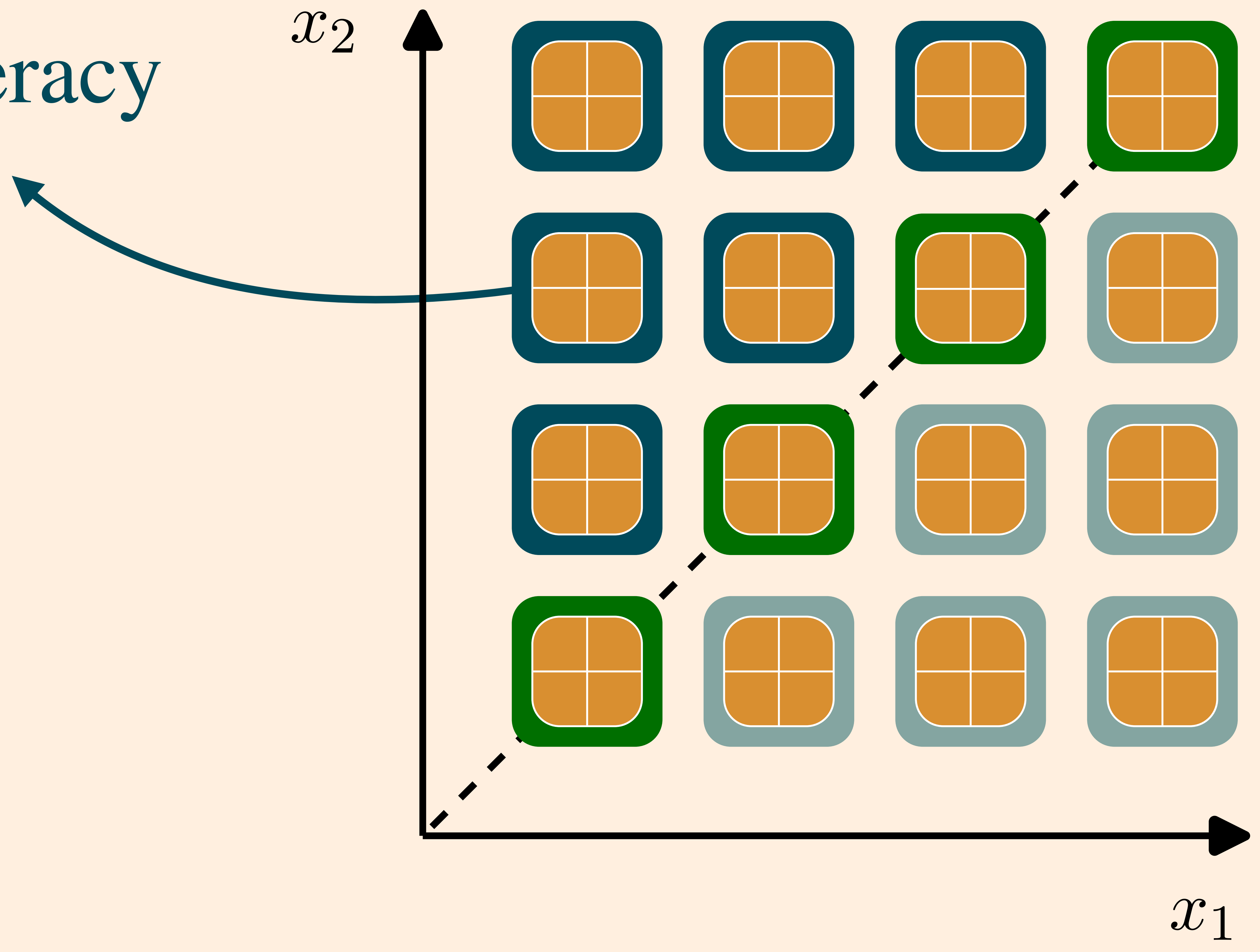
$x_1 < x_2$

$x_1 > x_2$

$x_1 = x_2$

$$|\Psi(\mathbf{k},\mathbf{s})\rangle = |\Psi\rangle_I + |\Psi\rangle_\Sigma + |\Psi\rangle_\partial$$

$\{A_{j,I}\}_{j:\text{degeneracy}}$



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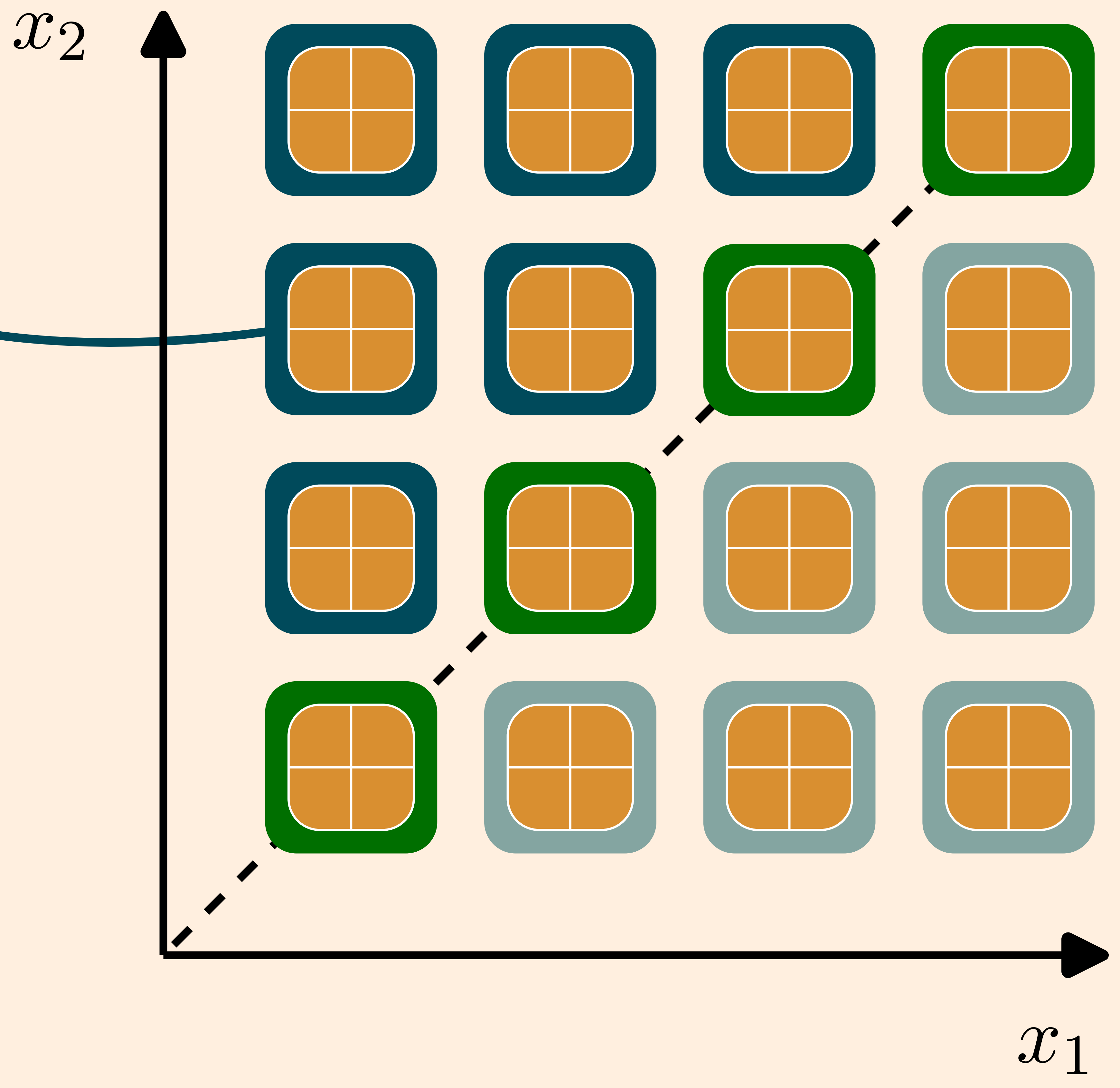


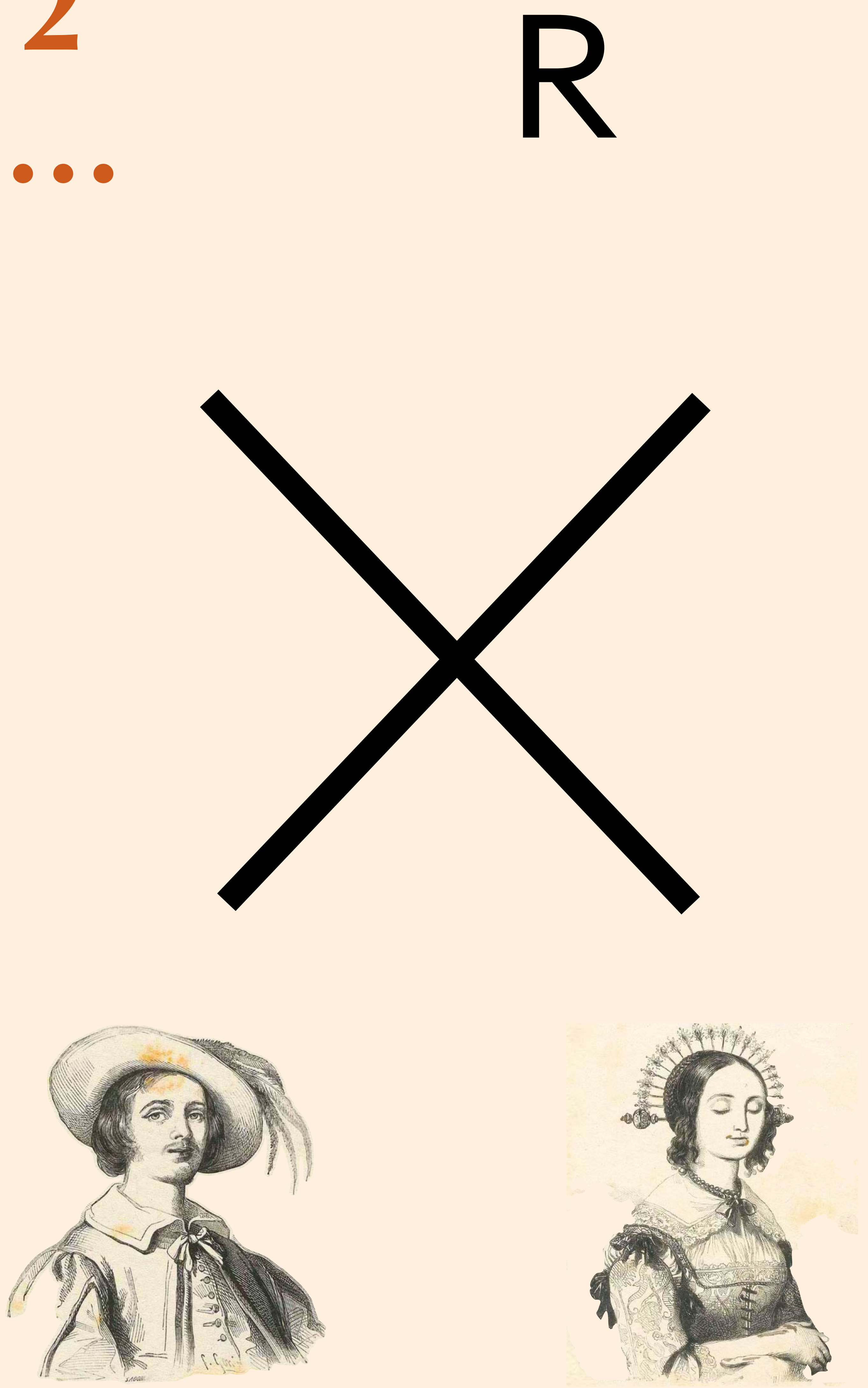
$x_1 < x_2$

$x_1 > x_2$

$x_1 = x_2$

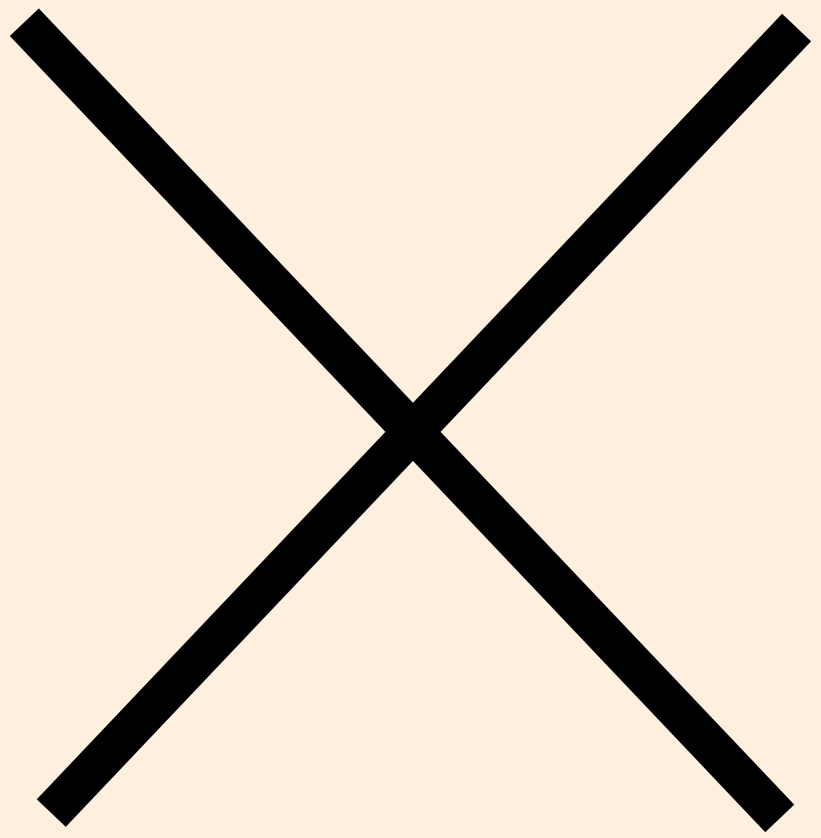
$N!2^{N-1}$ **coefficients!**

Solve for 2 particles …

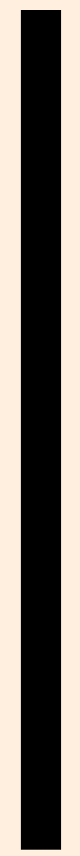


Solve for 2 particles …

$$R_{12} = R \otimes \mathbb{1}$$

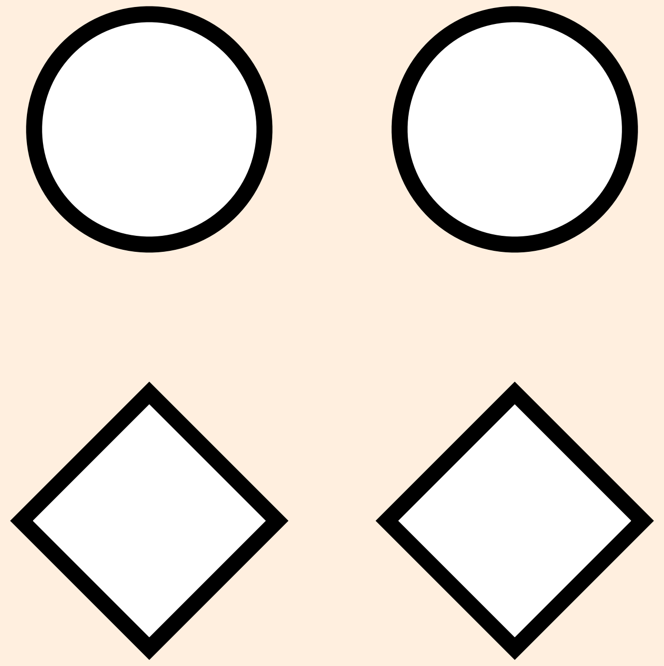


$\otimes$

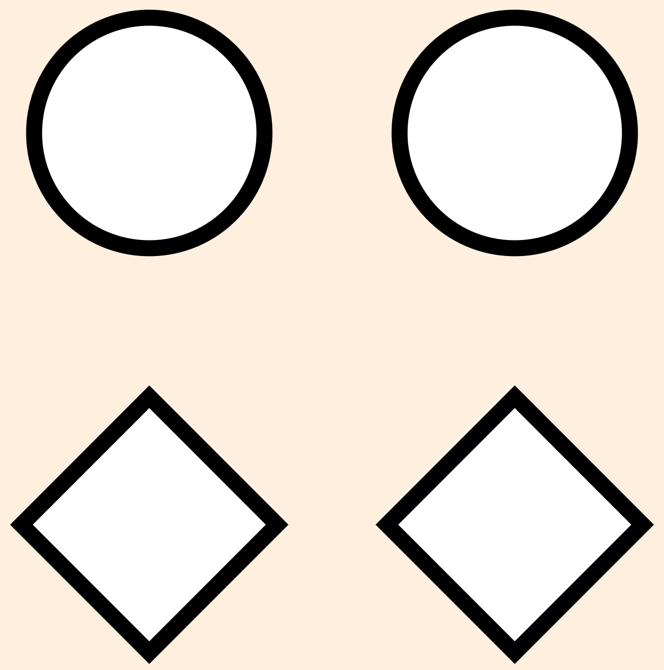


… and add a spectator

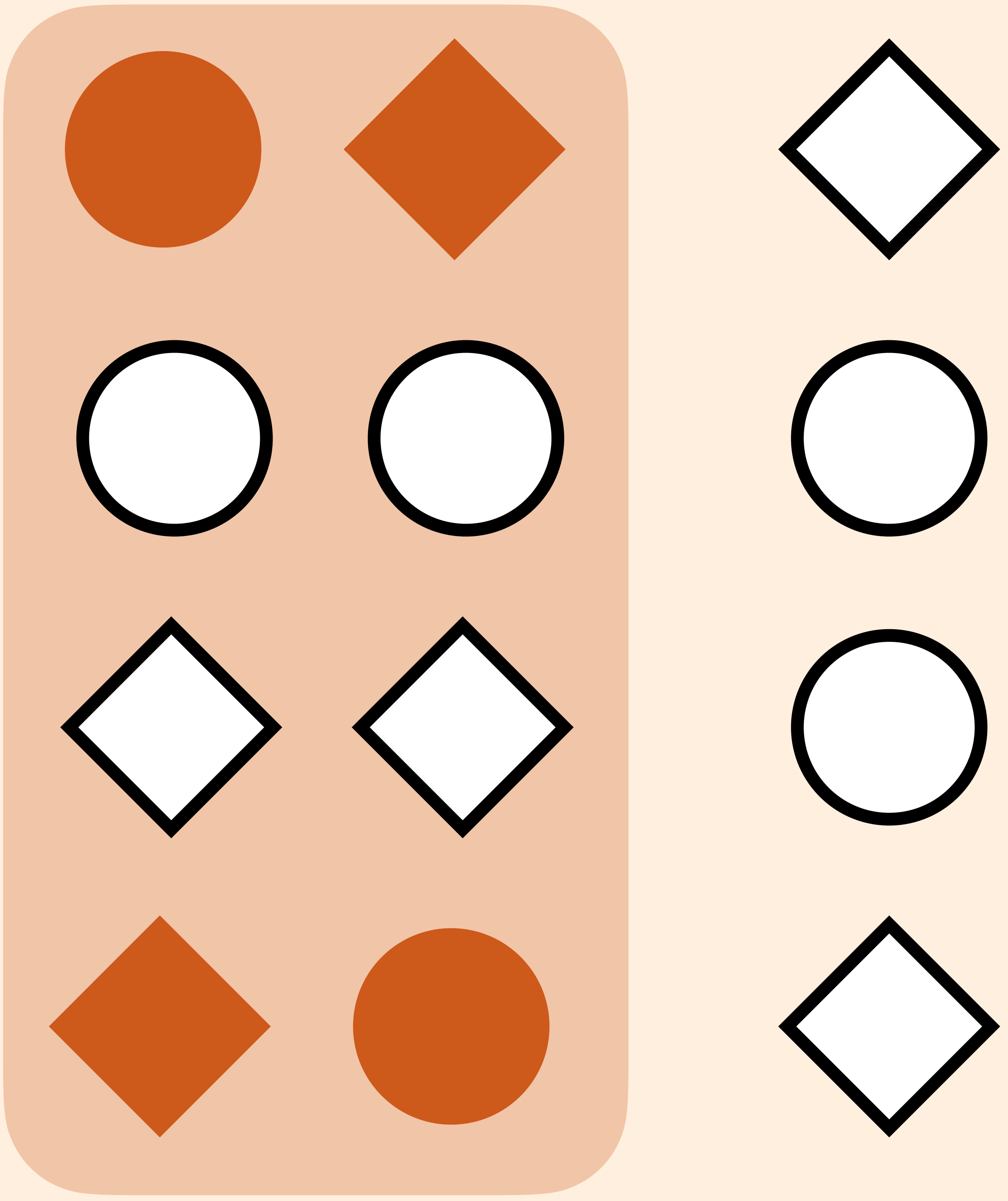
N=2



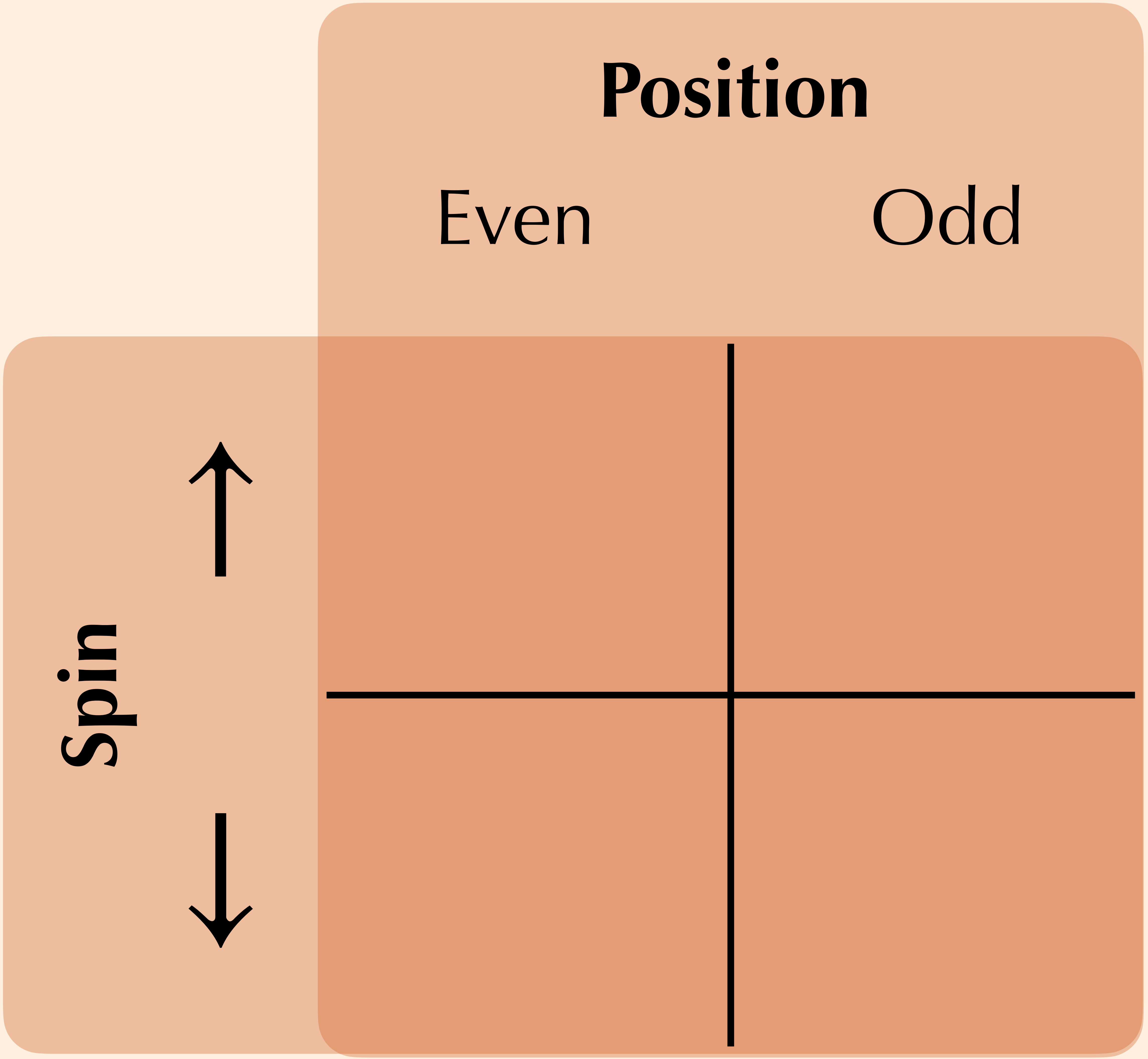
N=2

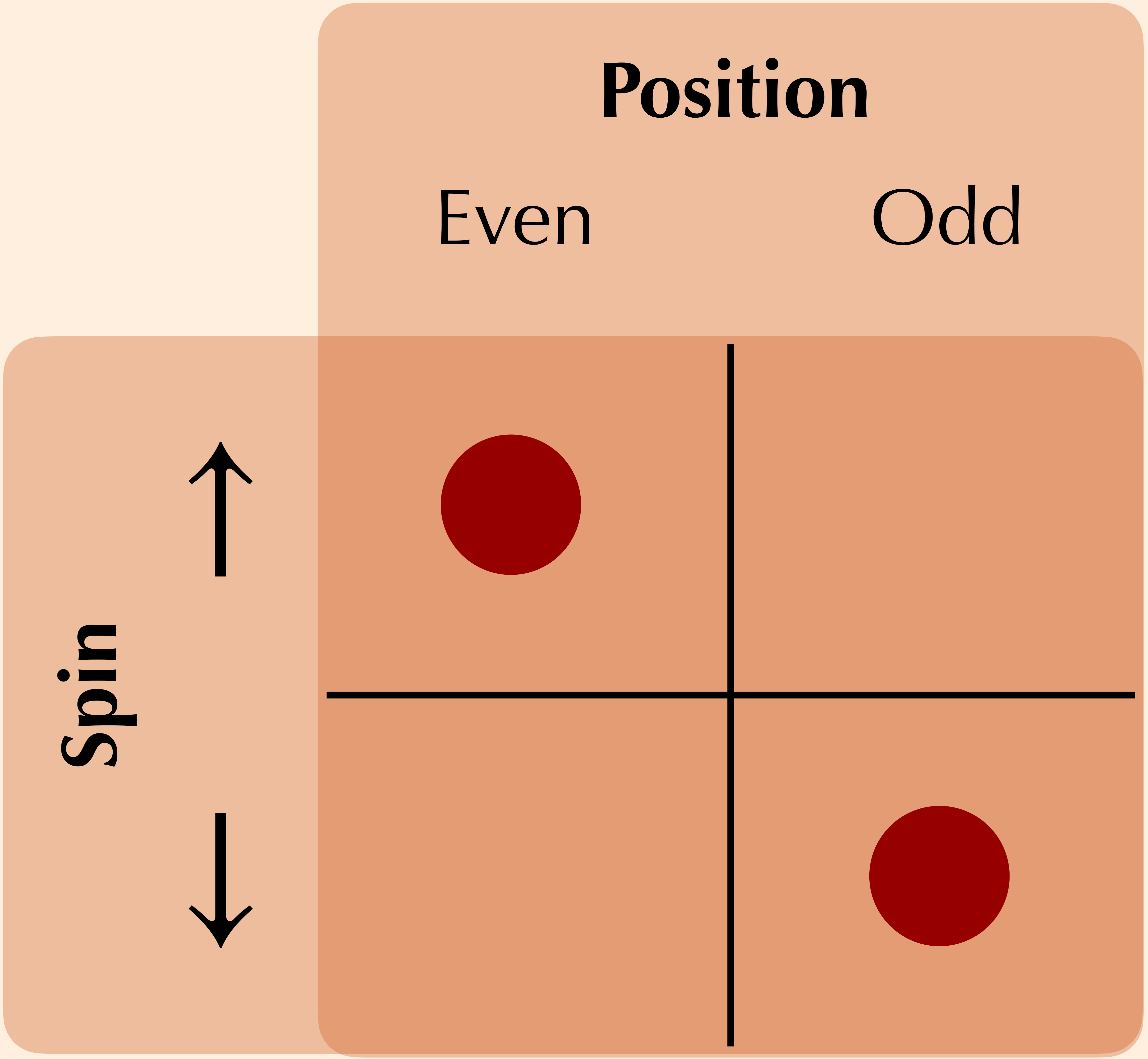


N=3



Solving the 2-particle sector is not enough!


Position
Even
Odd
Spin
↑
↓


Position
Even
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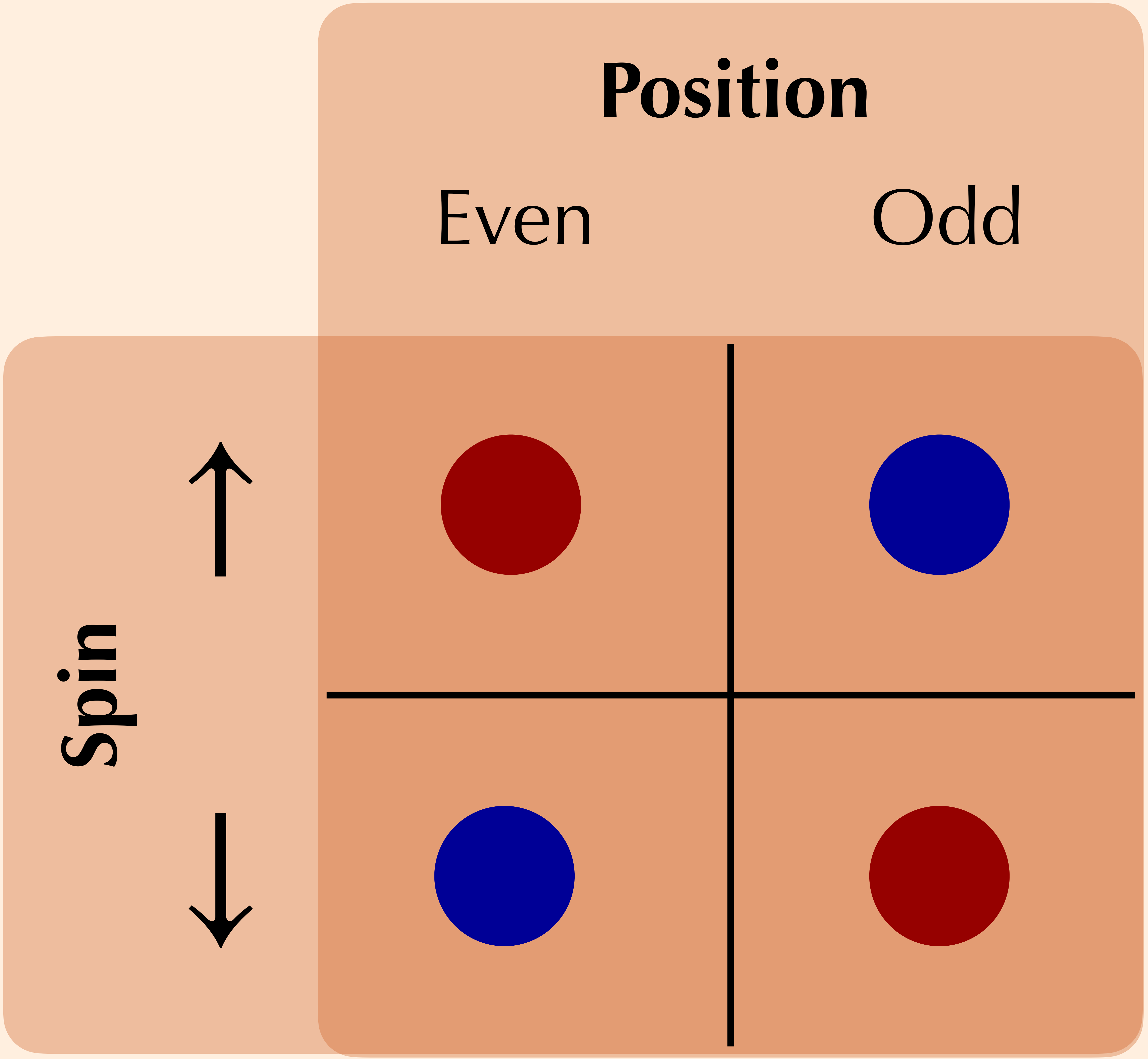
Position
Even
Odd
Spin
↑
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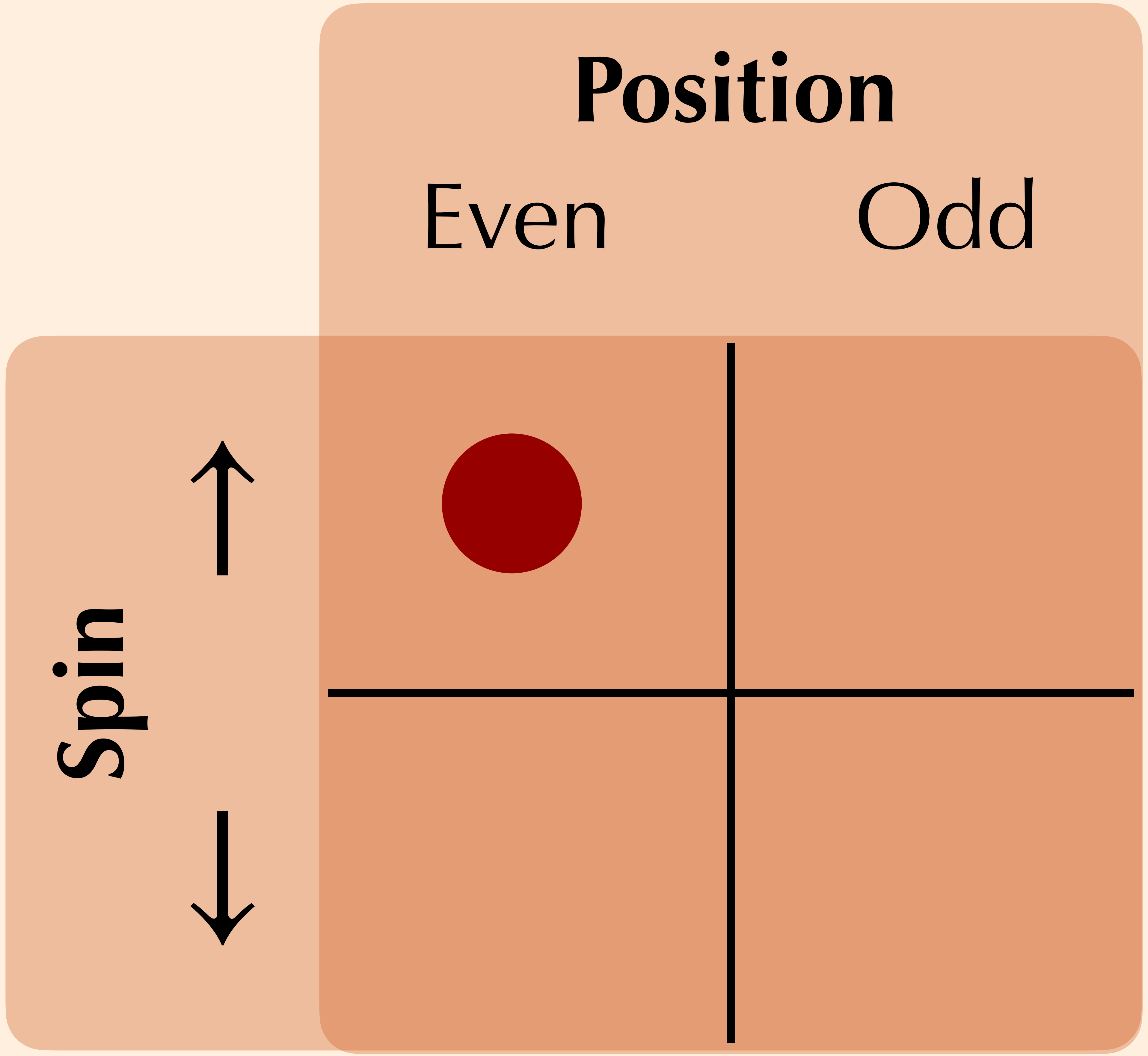
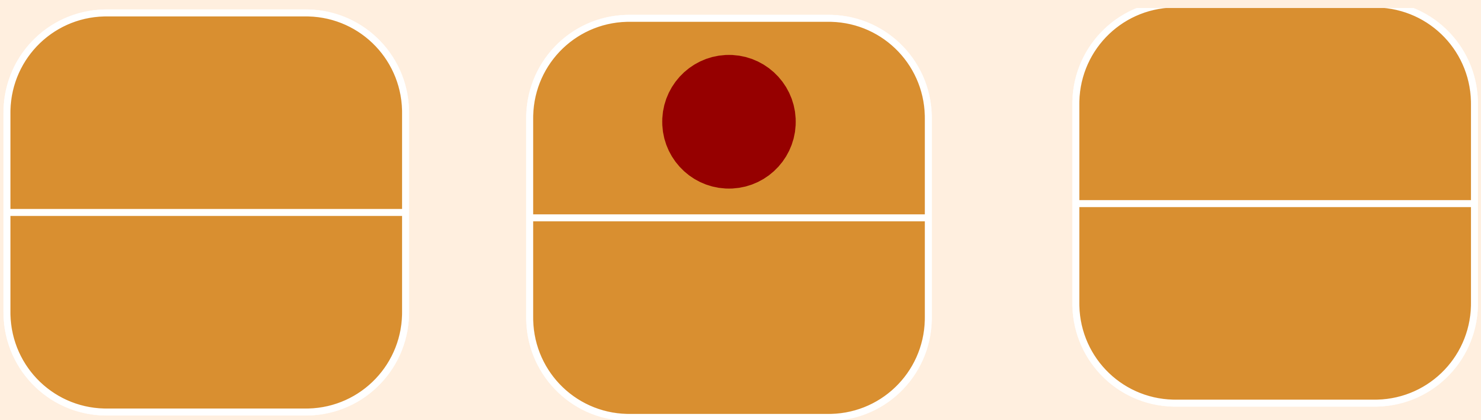
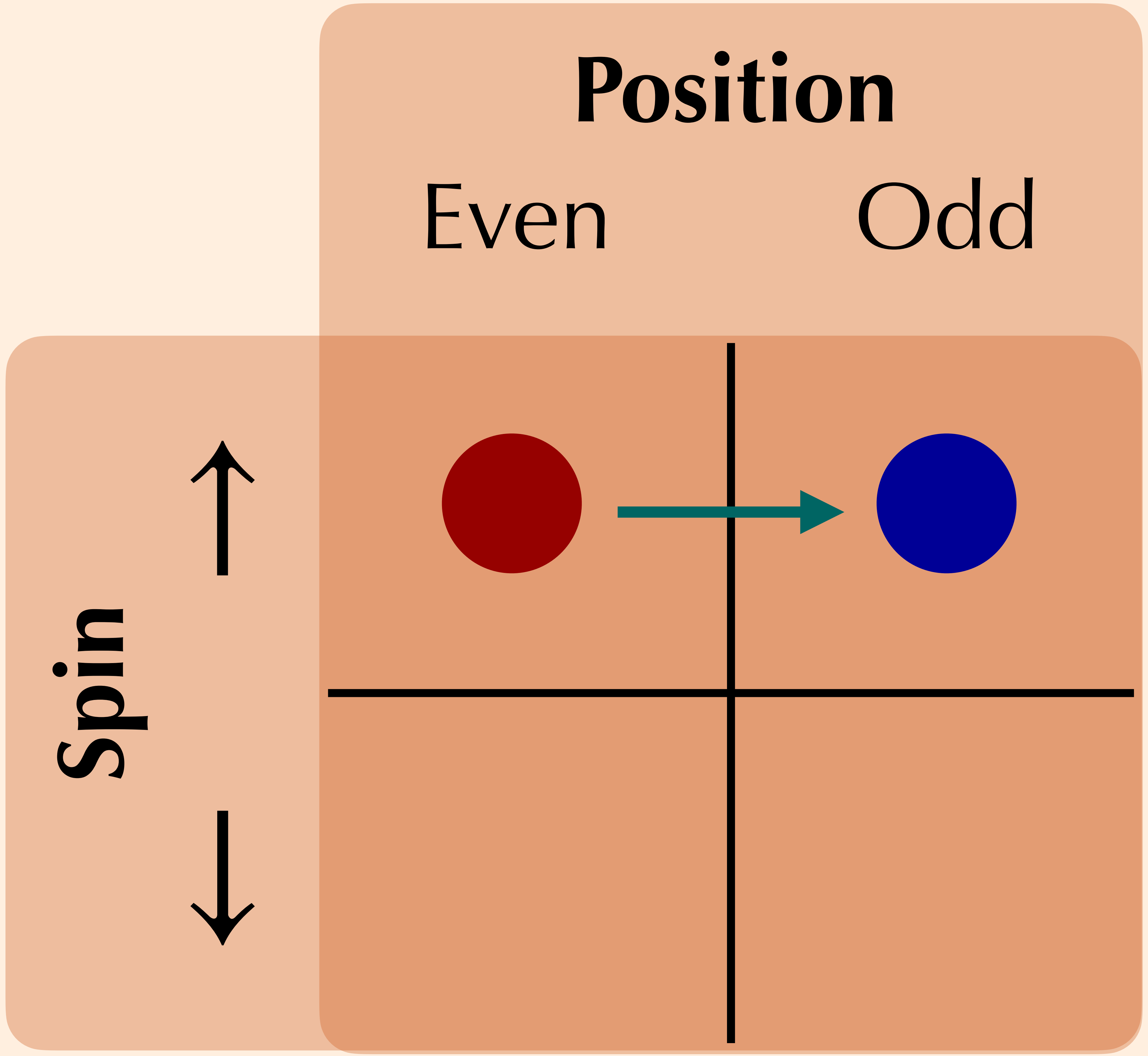
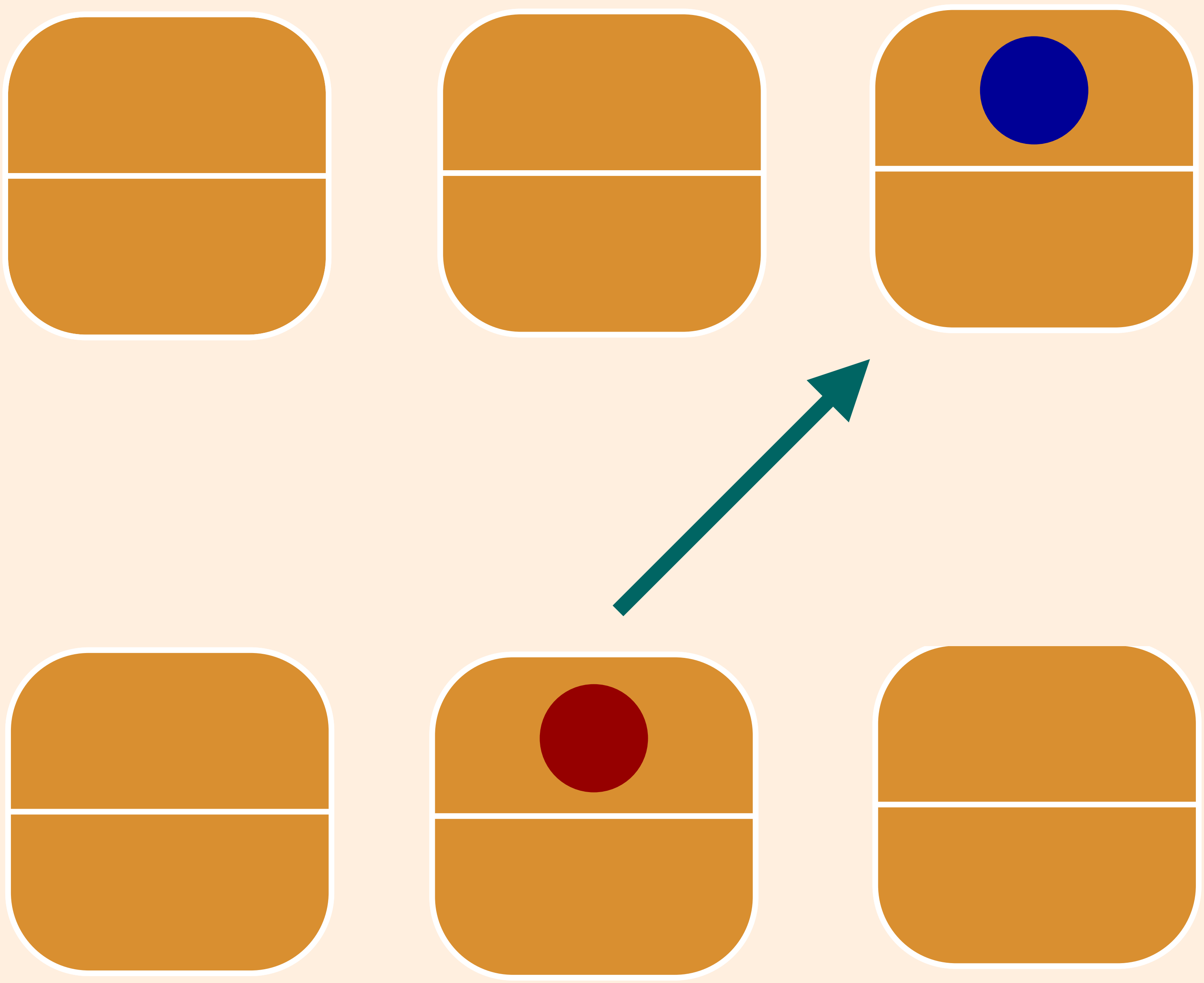


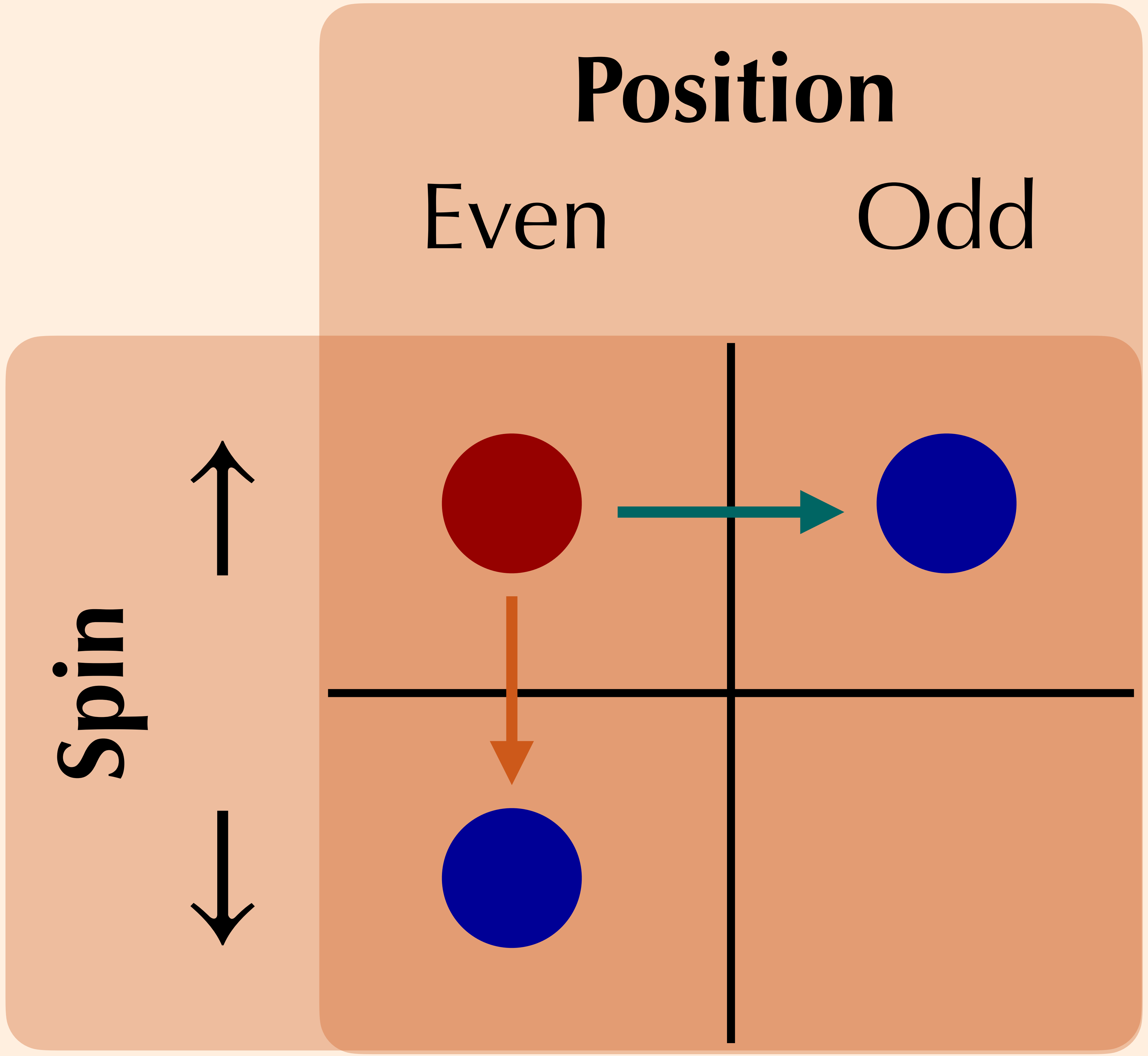


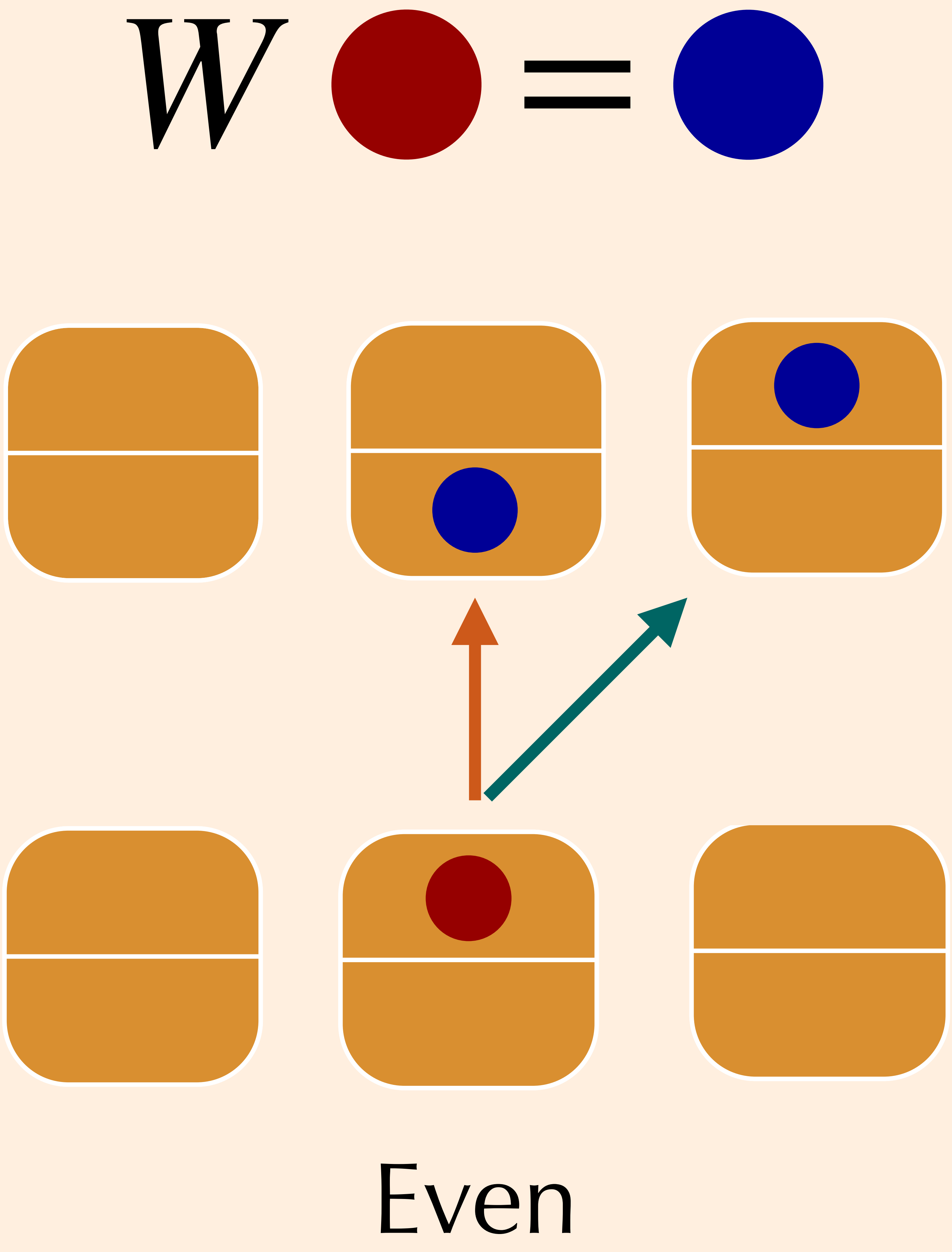
Even

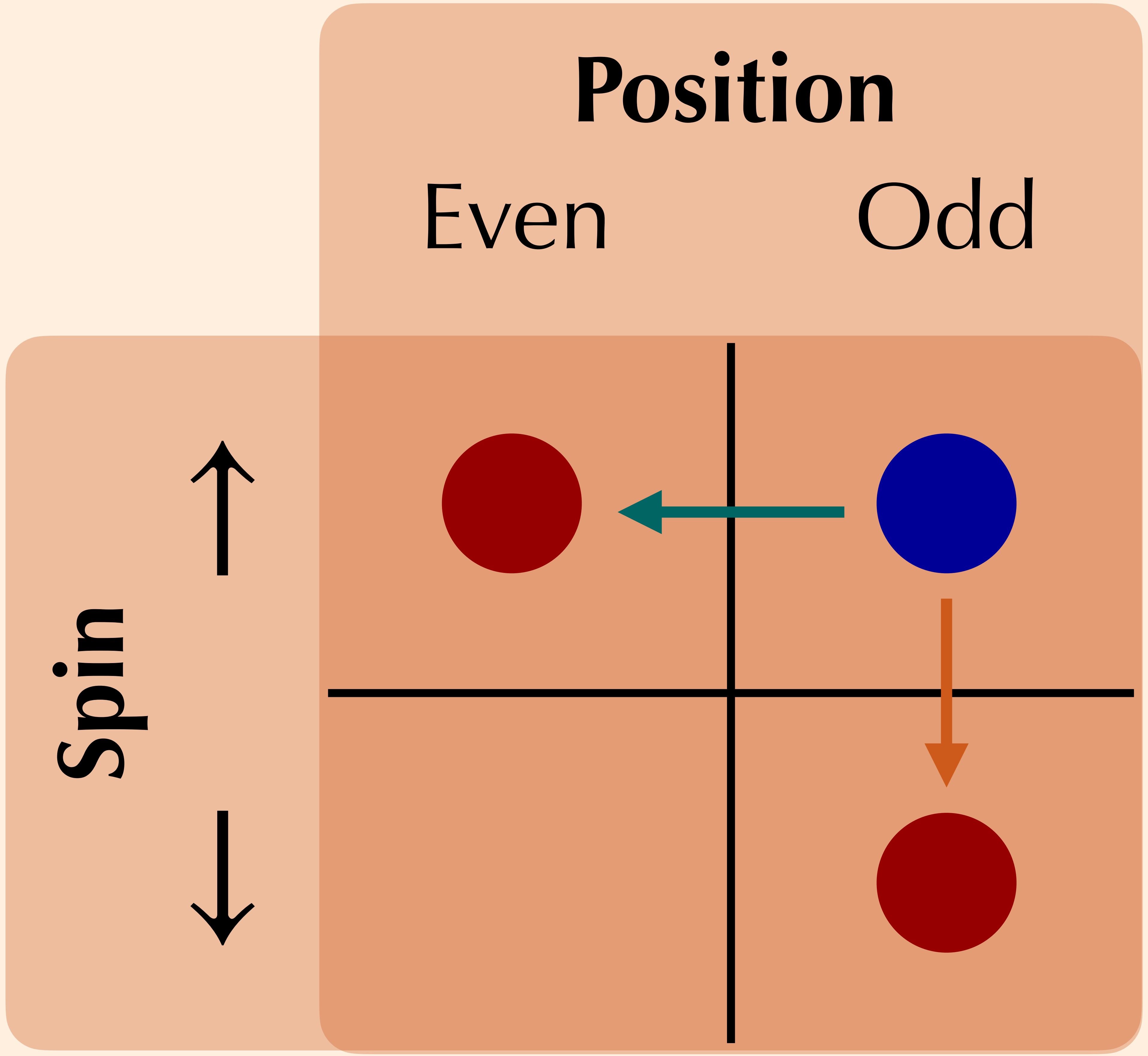


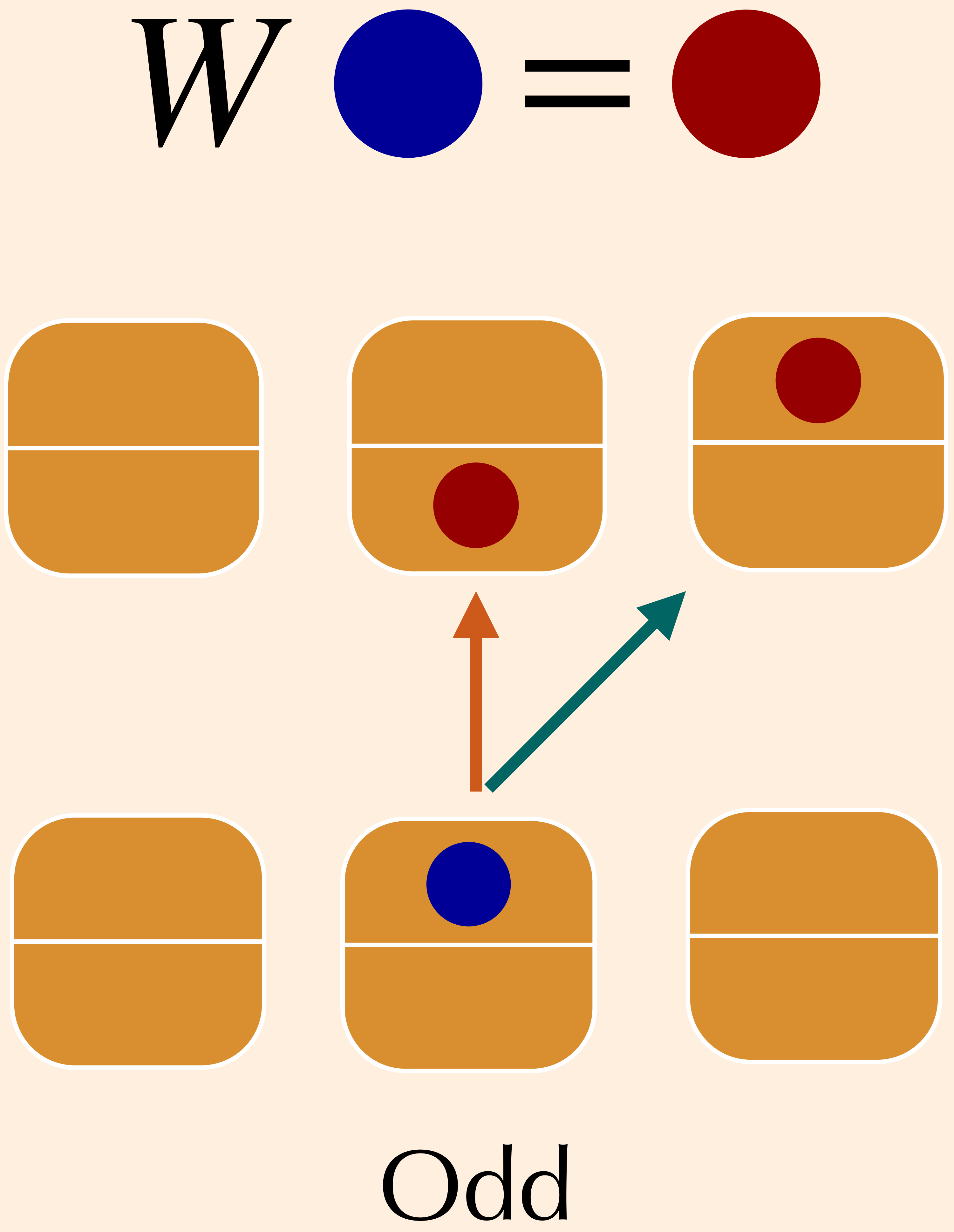


Even

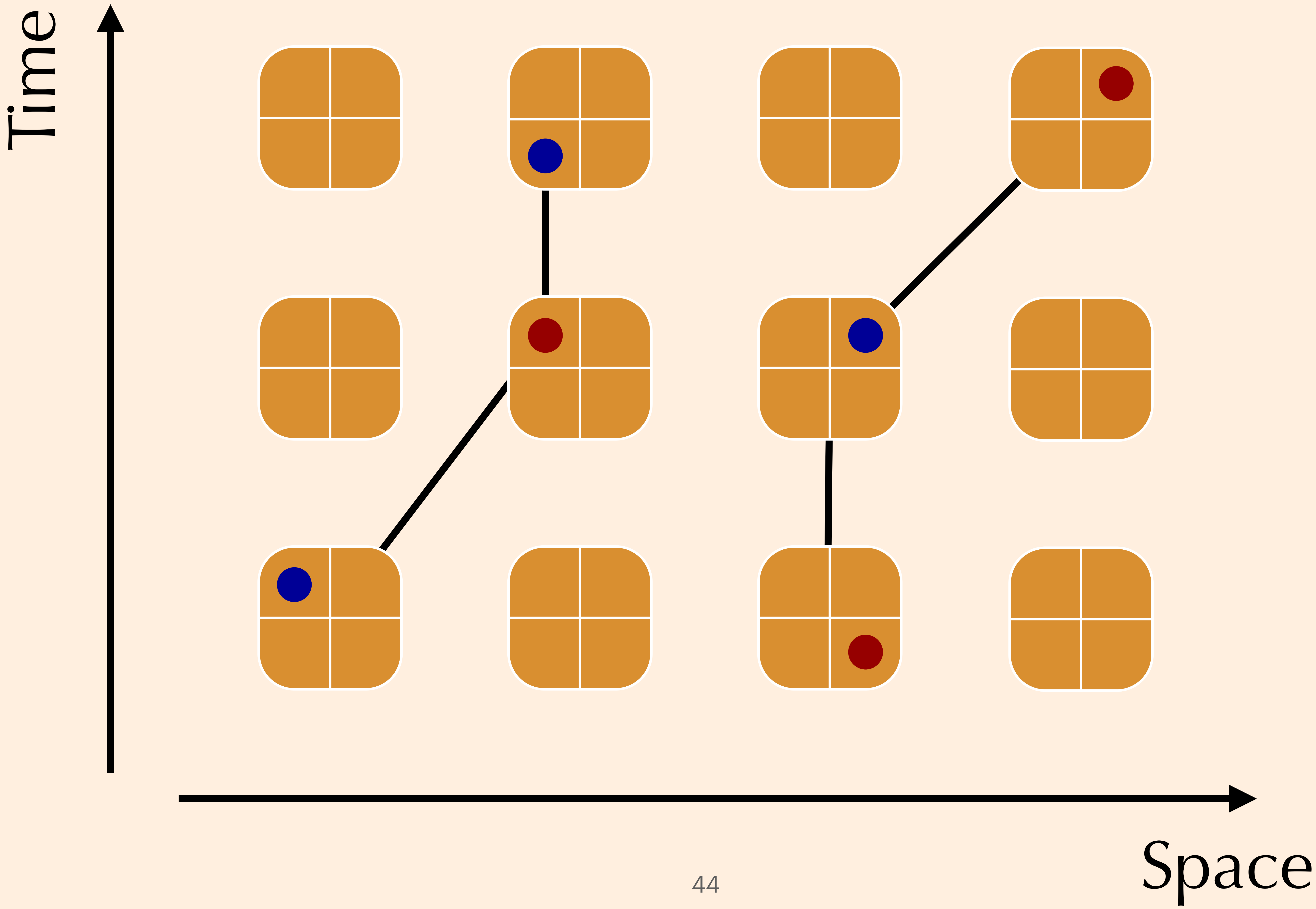

Position
Even
Odd
Spin
↑
↓

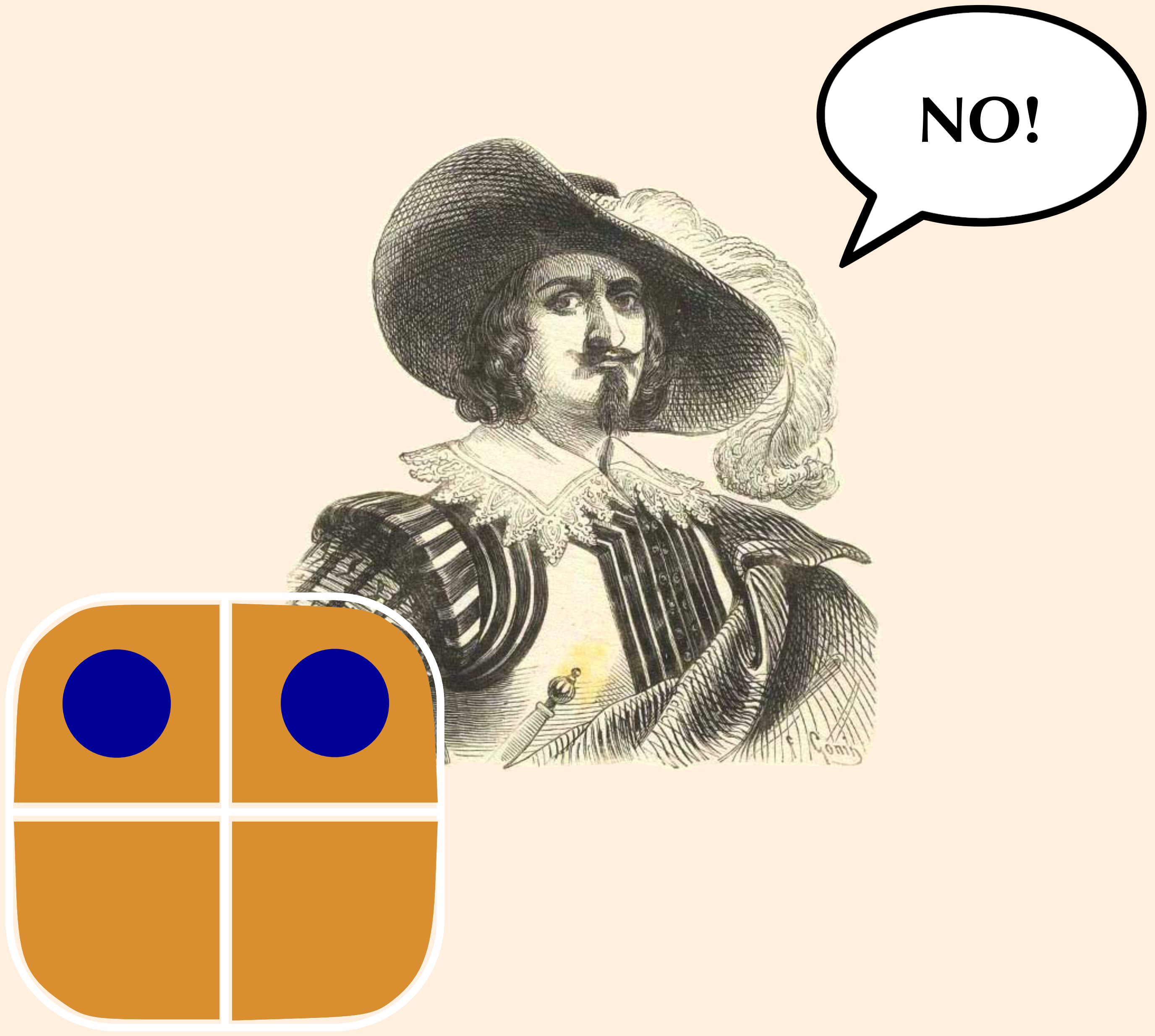

W
=
Even

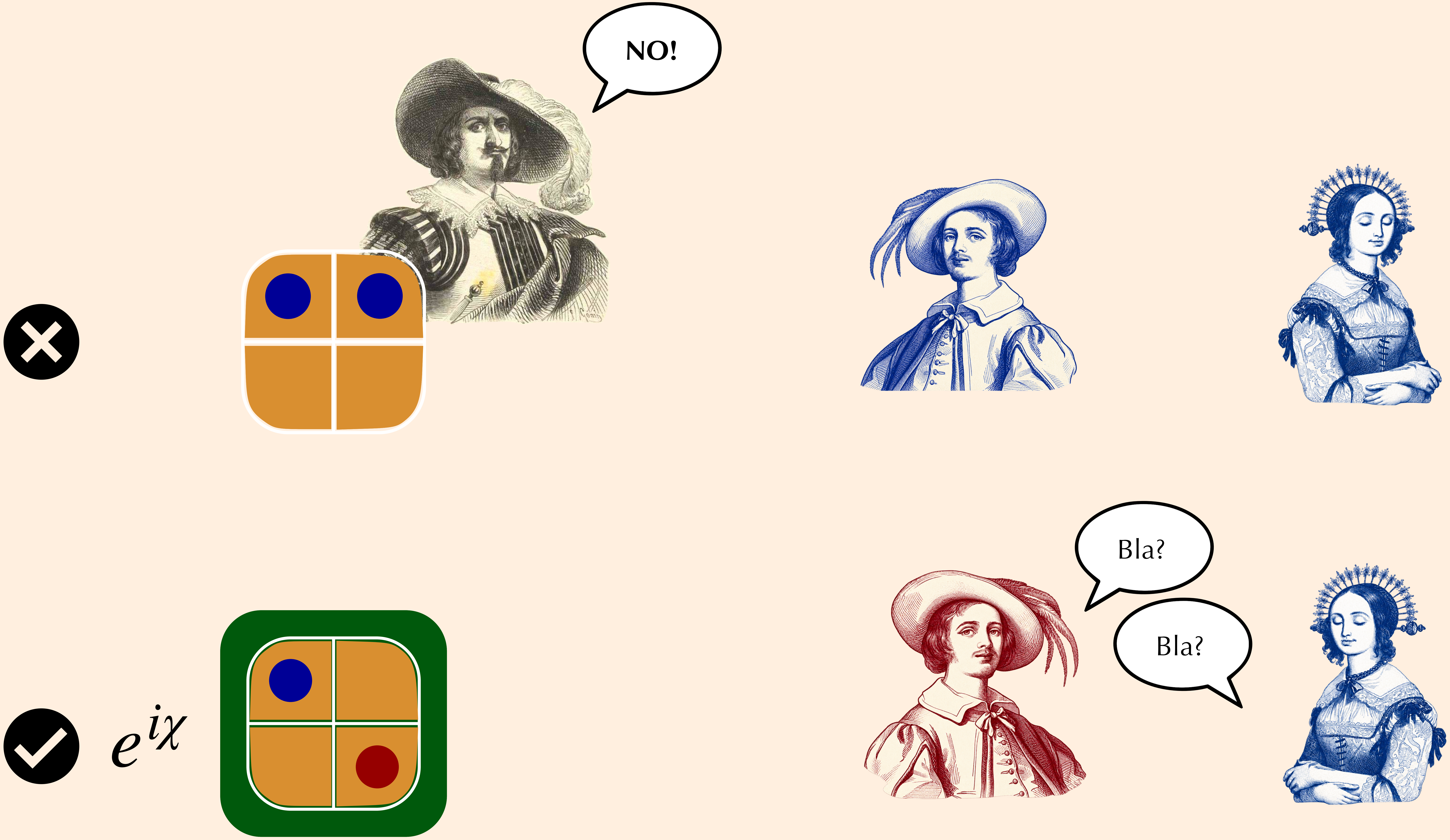

Position
Even
Odd
Spin
↑
↓


W
=
Odd

Relative colour is conserved!

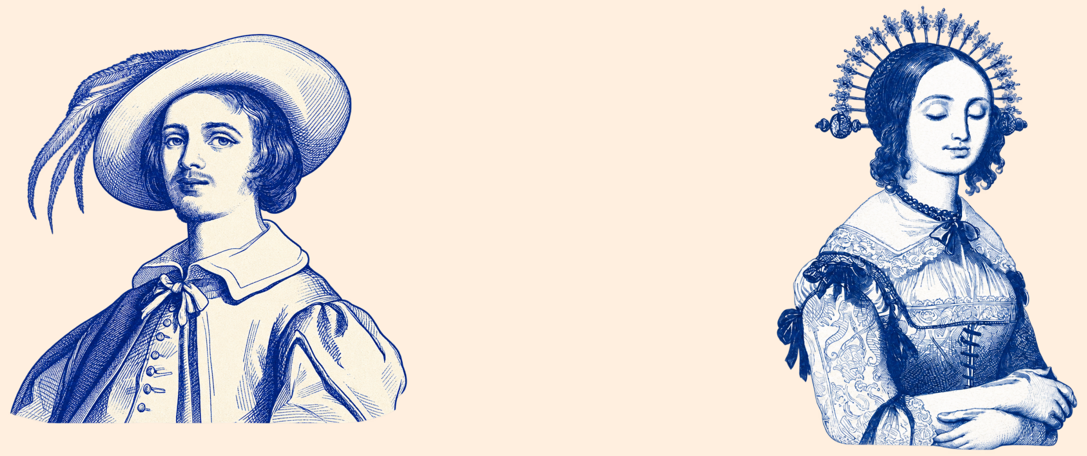



NO!

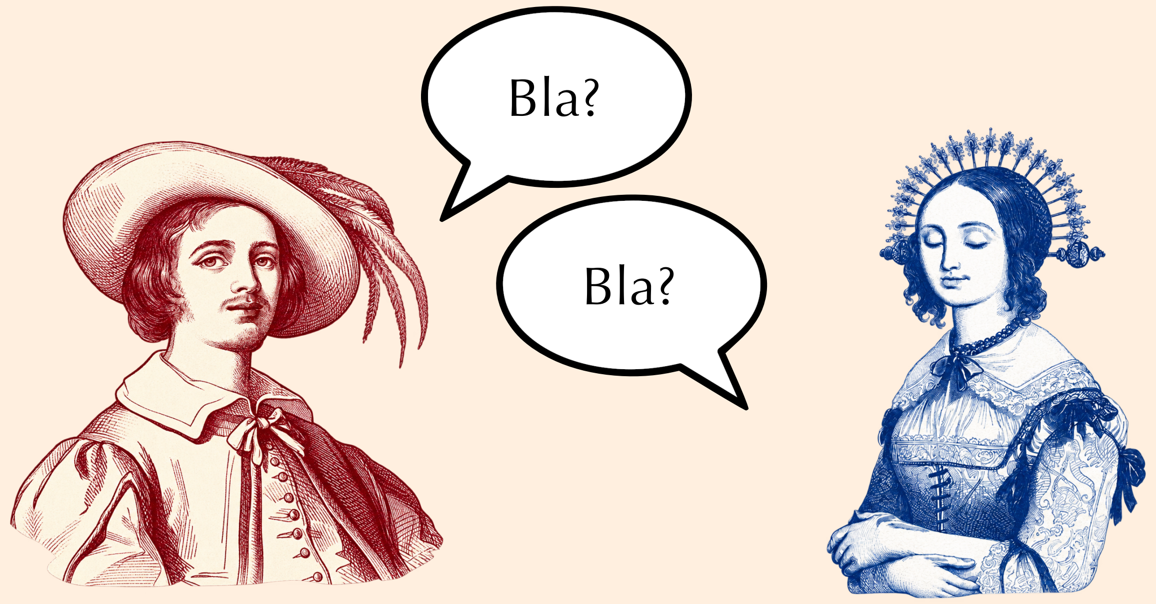

NO!
$e^{i\chi}$
Bla?
Bla?

Solve configurations with:

Same colours



Opposite colours



Spectator

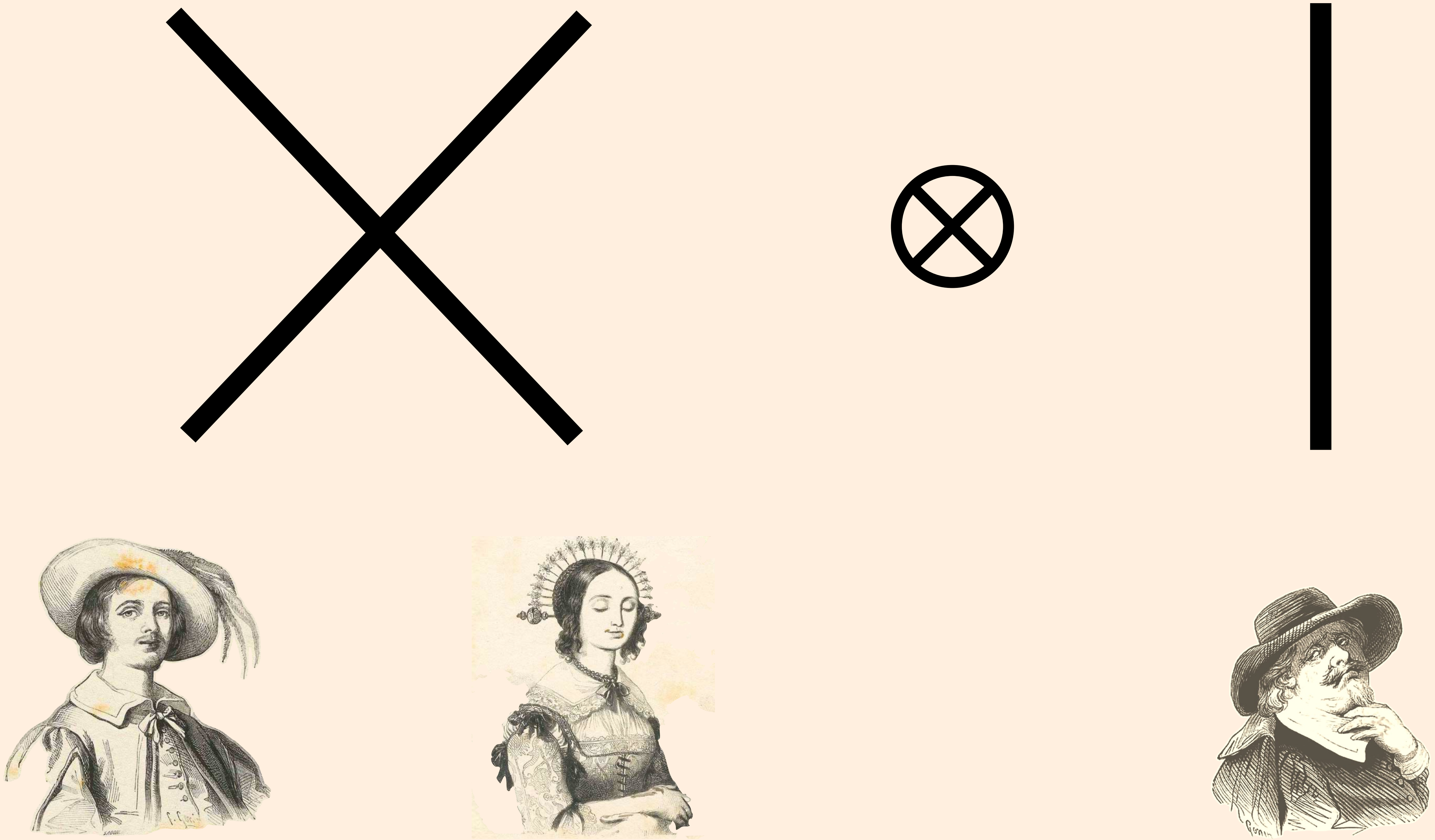
Relative colour R matrix

$$|=,=\rangle$$
$$|=,\neq\rangle$$
$$|\neq,=\rangle$$
$$|\neq,\neq\rangle$$

$$\mathsf{R}^{\mathrm{rel}}_{12}(u) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & a(u) & b(u) \\ & & b(u) & a(u) \end{bmatrix}$$

Not yet

$$R_{12} = R \otimes \mathbb{1}$$



Embedding E into **absolute colour** space

$$\text{e.g.}\quad E\,|\,=\,,\,\neq\,\rangle = \frac{1}{\sqrt{2}}\left[\,|\textcolor{red}{\bullet},\textcolor{red}{\bullet},\textcolor{blue}{\bullet}\rangle + |\textcolor{blue}{\bullet},\textcolor{blue}{\bullet},\textcolor{red}{\bullet}\rangle\right]$$

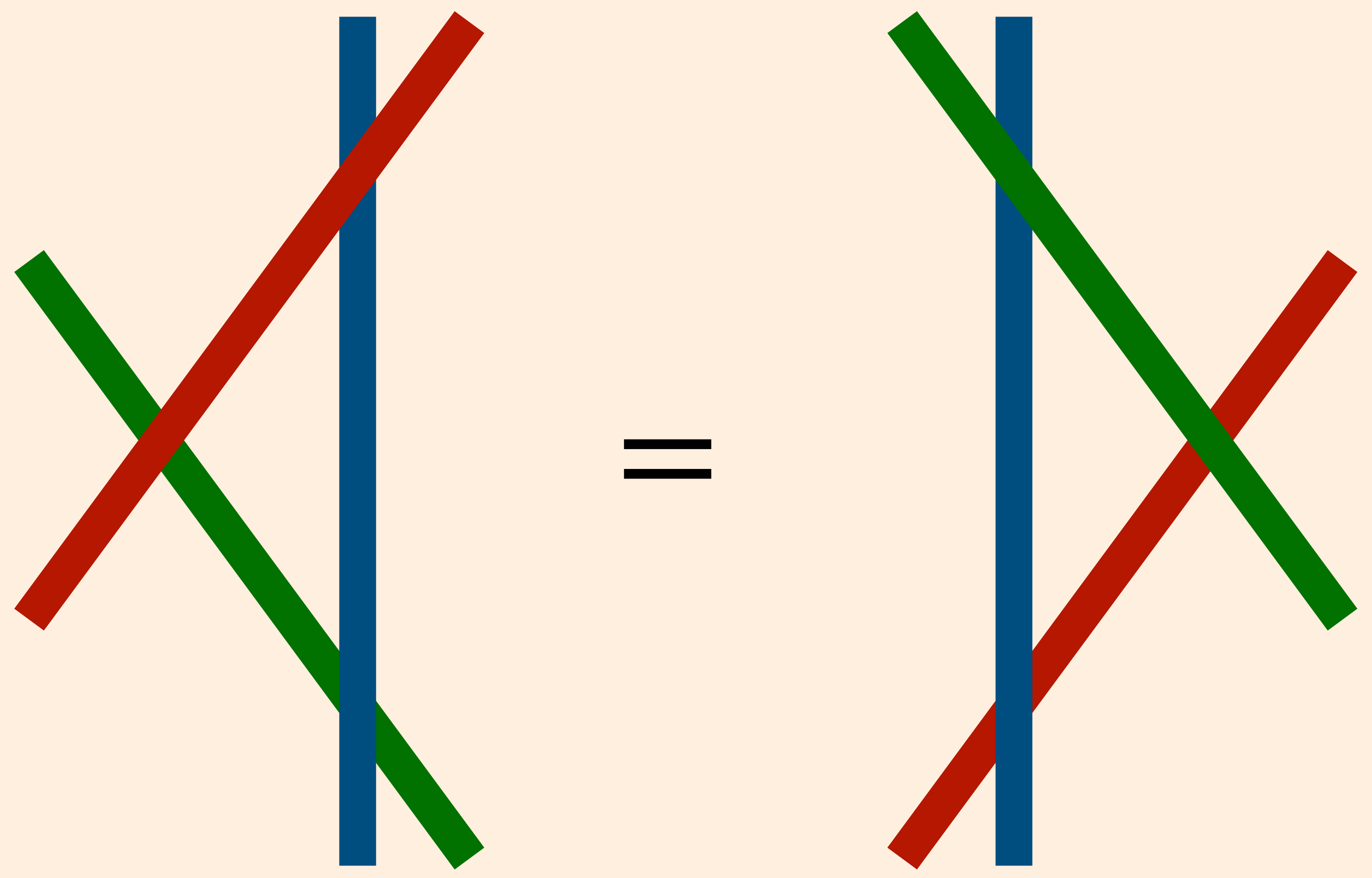
Embedding E into **absolute colour** space

e.g. $$E\,|\,=\,,\,\neq\,\rangle = \frac{1}{\sqrt{2}}\left[\,|{\color{red}\bullet},{\color{red}\bullet},{\color{blue}\bullet}\rangle + |{\color{blue}\bullet},{\color{blue}\bullet},{\color{red}\bullet}\rangle\right]$$

$$\mathsf{R}^{\mathrm{rel}}_{12}(u) \quad\hookrightarrow\quad \mathsf{R}^{\mathrm{abs}}_{12}(u) = {\color{orange}\mathsf{R}} \otimes \square$$

6-vertex model R matrix

$$\mathsf{R}(u) = \begin{pmatrix} 1 & & & \\ & a(u) & b(u) & \\ & b(u) & a(u) & \\ & & & 1 \end{pmatrix}$$



Absolute colours reveal a known scattering structure.

In the *same* ordered basis

$\mathsf{R}^{\mathrm{rel}}_{12}$

$$\mathsf{R}^{\mathrm{abs}}_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 & & & & \\ 0 & 1 & 0 & 0 & & & & \\ 0 & 0 & a_{12} & b_{12} & & & & \\ 0 & 0 & b_{12} & a_{12} & & & & \\ & & & & 1 & 0 & 0 & 0 \\ & & & & 0 & 1 & 0 & 0 \\ & & & & 0 & 0 & a_{12} & -b_{12} \\ & & & & 0 & 0 & -b_{12} & a_{12} \end{bmatrix}$$

$\mathsf{R}^{\mathrm{rel}}_{23}$

$$\mathsf{R}^{\mathrm{abs}}_{23} = \begin{bmatrix} 1 & 0 & 0 & 0 & & & & \\ 0 & a_{23} & 0 & b_{23} & & & & \\ 0 & 0 & 1 & 0 & & & & \\ 0 & b_{23} & 0 & a_{23} & & & & \\ & & & & 1 & 0 & 0 & 0 \\ & & & & 0 & a_{23} & 0 & b_{23} \\ & & & & 0 & 0 & 1 & 0 \\ & & & & 0 & b_{23} & 0 & a_{23} \end{bmatrix}$$

$\mathsf{R}^{\mathrm{rel}}_{13}$

$$\mathsf{R}^{\mathrm{abs}}_{13} = \begin{bmatrix} 1 & 0 & 0 & 0 & & & & \\ 0 & a_{13} & b_{13} & 0 & & & & \\ 0 & b_{13} & a_{13} & 0 & & & & \\ 0 & 0 & 0 & 1 & & & & \\ & & & & 1 & 0 & 0 & 0 \\ & & & & 0 & a_{13} & -b_{13} & 0 \\ & & & & 0 & -b_{13} & a_{13} & 0 \\ & & & & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \mathsf{R}^{\mathrm{rel}}_{12}\,\mathsf{R}^{\mathrm{rel}}_{13}\,\mathsf{R}^{\mathrm{rel}}_{23} = \mathsf{R}^{\mathrm{rel}}_{23}\,\mathsf{R}^{\mathrm{rel}}_{13}\,\mathsf{R}^{\mathrm{rel}}_{12}$$

In the *same* ordered basis

$$\mathsf{R}^{\mathrm{rel}}_{12}$$

$$\mathsf{R}^{\mathrm{abs}}_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 & & & & \\ 0 & 1 & 0 & 0 & & & & \\ 0 & 0 & a_{12} & b_{12} & & & & \\ 0 & 0 & b_{12} & a_{12} & & & & \\ & & & & 1 & 0 & 0 & 0 \\ & & & & 0 & 1 & 0 & 0 \\ & & & & 0 & 0 & a_{12} & -b_{12} \\ & & & & 0 & 0 & -b_{12} & a_{12} \end{bmatrix}$$

$$\mathsf{R}^{\mathrm{rel}}_{23}$$

$$\mathsf{R}^{\mathrm{abs}}_{23} = \begin{bmatrix} 1 & 0 & 0 & 0 & & & & \\ 0 & a_{23} & 0 & b_{23} & & & & \\ 0 & 0 & 1 & 0 & & & & \\ 0 & b_{23} & 0 & a_{23} & & & & \\ & & & & 1 & 0 & 0 & 0 \\ & & & & 0 & a_{23} & 0 & b_{23} \\ & & & & 0 & 0 & 1 & 0 \\ & & & & 0 & b_{23} & 0 & a_{23} \end{bmatrix}$$

$$\mathsf{R}^{\mathrm{rel}}_{13}$$

$$\mathsf{R}^{\mathrm{abs}}_{13} = \begin{bmatrix} 1 & 0 & 0 & 0 & & & & \\ 0 & a_{13} & b_{13} & 0 & & & & \\ 0 & b_{13} & a_{13} & 0 & & & & \\ 0 & 0 & 0 & 1 & & & & \\ & & & & 1 & 0 & 0 & 0 \\ & & & & 0 & a_{13} & -b_{13} & 0 \\ & & & & 0 & -b_{13} & a_{13} & 0 \\ & & & & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \mathsf{R}_{12}\,\mathsf{R}_{13}\,\mathsf{R}_{23} = \mathsf{R}_{23}\,\mathsf{R}_{13}\,\mathsf{R}_{12}$$

π-degeneracy space

π-degeneracy space

○ ◇

$\xrightarrow{\text{Change of basis}}$

Relative colour space

$=$, $\neq$

π-degeneracy space

○ ◇

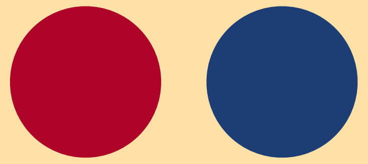
$$\xrightarrow{\text{Change of basis}}$$

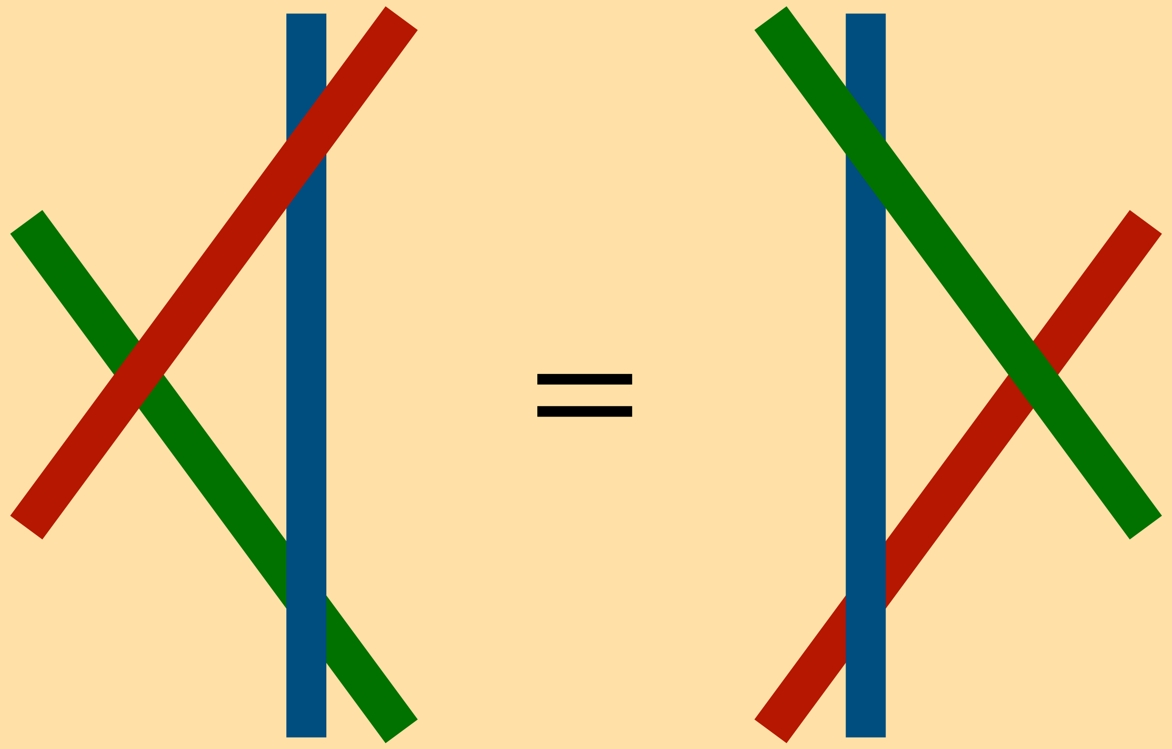
Relative colour space

$= \, , \, \neq$

$$\xhookrightarrow{\text{Embedding}}$$

Absolute colour space





Known integrable structure

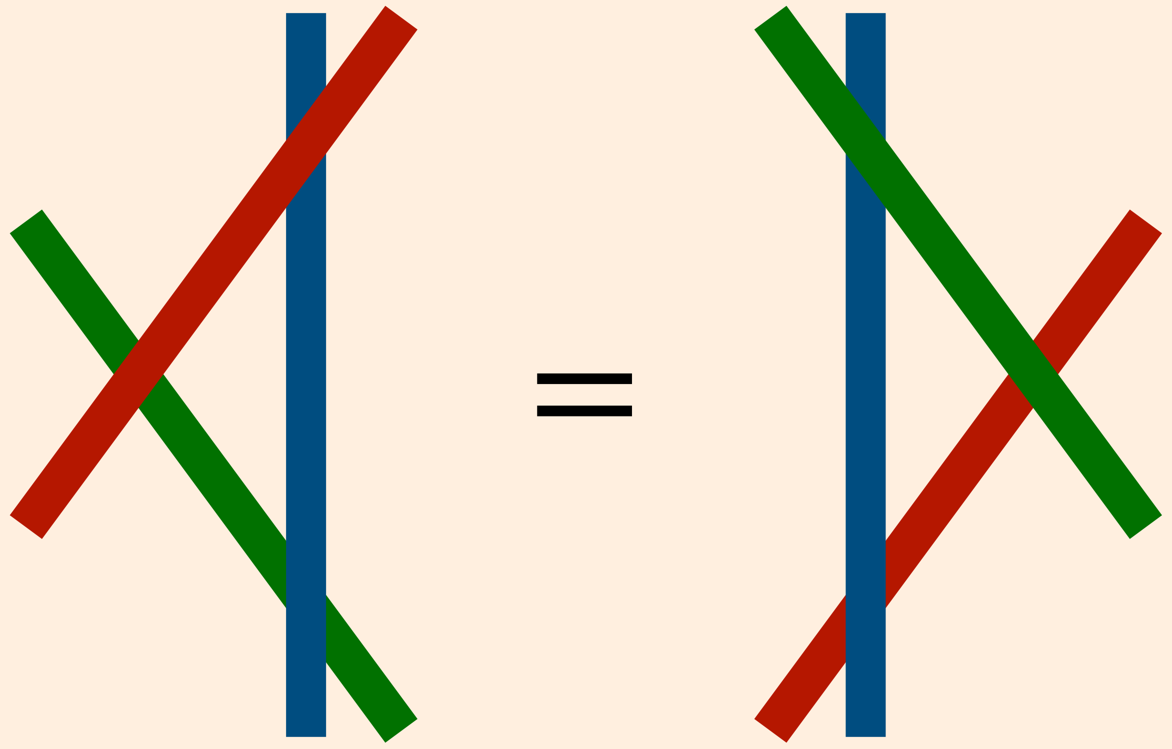
π-degeneracy space

○ ◇

$\xrightarrow{\text{Change of basis}}$

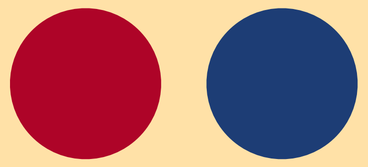
Relative colour space

$= , \neq$

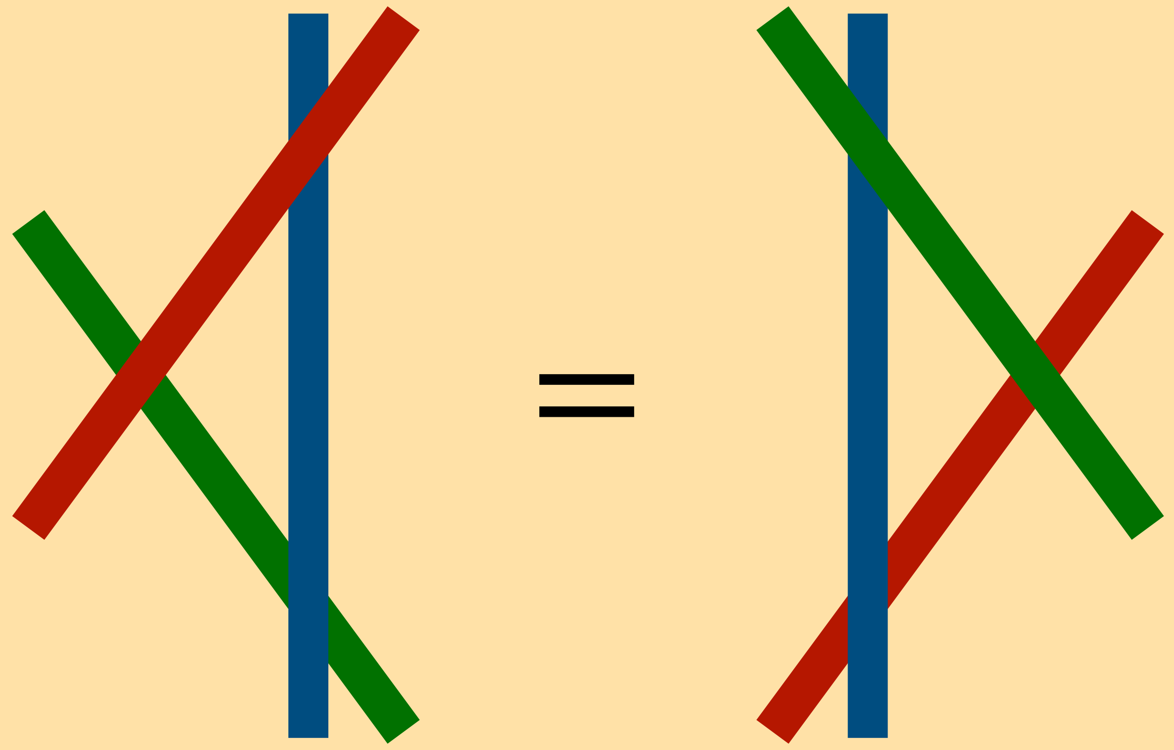


$\xhookrightarrow{\text{Embedding}}$

Absolute colour space

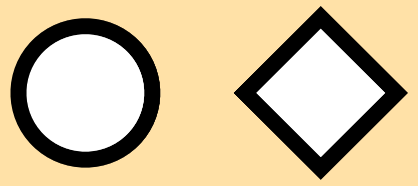


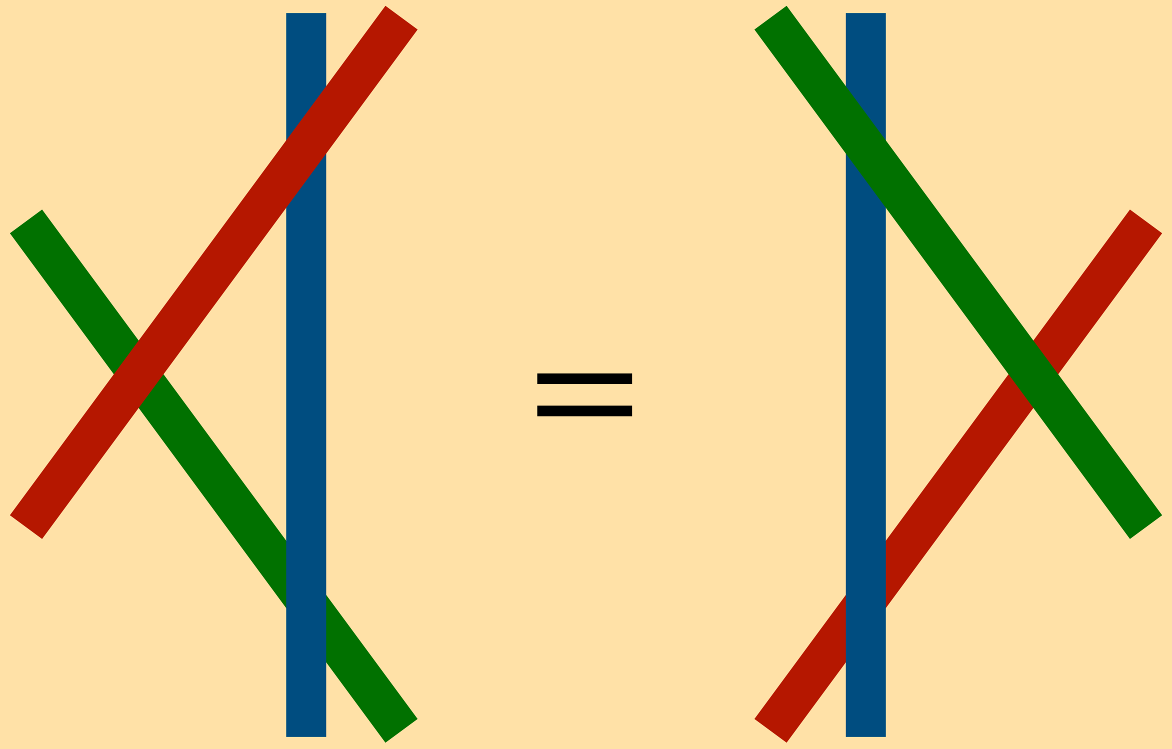
$\longleftarrow$



Known integrable structure

π-degeneracy space





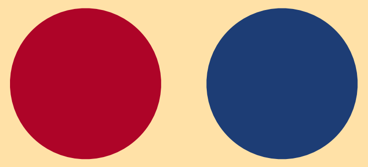
$\xrightarrow{\text{Change of basis}}$

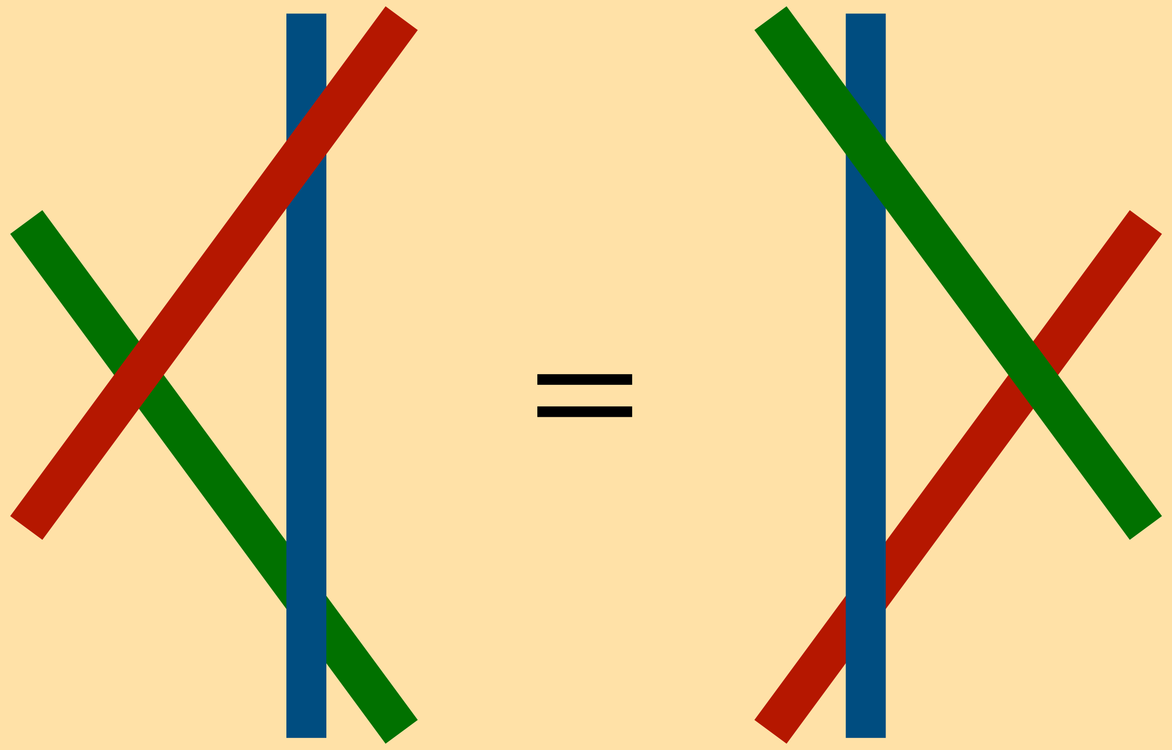
Relative colour space

$= \, , \neq$

$\xhookrightarrow{\text{Embedding}}$

Absolute colour space





Known integrable structure

Can many-particle scattering be built *consistently* from two-particle collisions?

Yes, but…

… is the solution set **complete**?

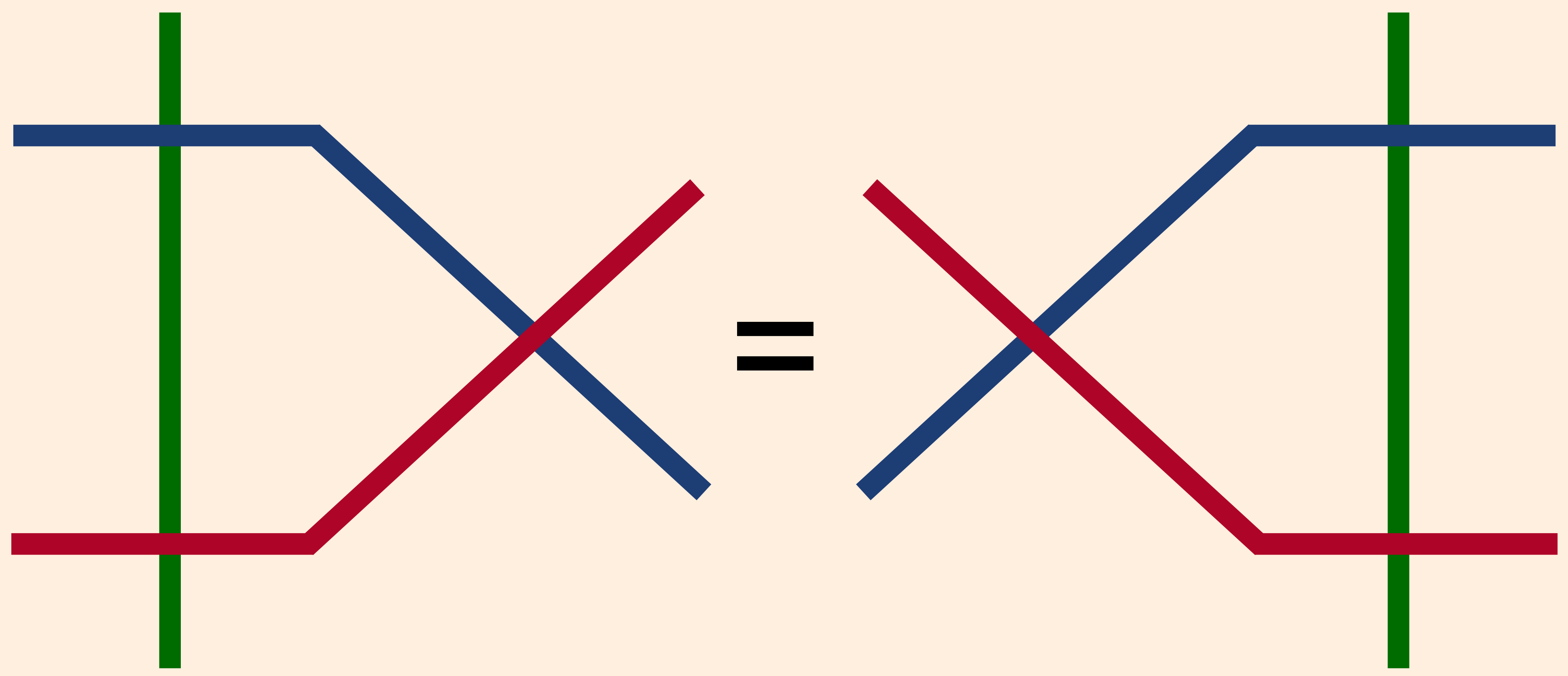
… does it include **bound states**?

Yes, but…

… is the solution set **complete**?

… does it include **bound states**?

Algebraic Bethe Ansatz?



$$|\,\Psi(\mathbf{k},\mathbf{s})\rangle_I = \sum_{\mathbf{x}\in\mathscr{D}_I}\left\{A^{00}_{I,I}e^{i\langle\mathbf{k}|\mathbf{x}\rangle}\begin{pmatrix}\Phi_{\uparrow\uparrow}(\mathbf{a})\\ \Phi_{\uparrow\downarrow}(\mathbf{a})\\ \Phi_{\downarrow\uparrow}(\mathbf{a})\\ \Phi_{\downarrow\downarrow}(\mathbf{a})\end{pmatrix} + (-1)^{x_1+x_2}A^{11}_{I,I}e^{i\langle\mathbf{k}|\mathbf{x}\rangle}\begin{pmatrix}\Phi_{\uparrow\uparrow}(\mathbf{a})\\ -\Phi_{\uparrow\downarrow}(\mathbf{a})\\ -\Phi_{\downarrow\uparrow}(\mathbf{a})\\ \Phi_{\downarrow\downarrow}(\mathbf{a})\end{pmatrix} + A^{00}_{\Sigma,I}e^{i\langle\Sigma\mathbf{k}|\mathbf{x}\rangle}\begin{pmatrix}\Phi_{\uparrow\uparrow}(\mathbf{a})\\ \Phi_{\downarrow\uparrow}(\mathbf{a})\\ \Phi_{\uparrow\downarrow}(\mathbf{a})\\ \Phi_{\downarrow\downarrow}(\mathbf{a})\end{pmatrix} + (-1)^{x_1+x_2}A^{11}_{\Sigma,I}e^{i\langle\Sigma\mathbf{k}|\mathbf{x}\rangle}\begin{pmatrix}\Phi_{\uparrow\uparrow}(\mathbf{a})\\ -\Phi_{\downarrow\uparrow}(\mathbf{a})\\ -\Phi_{\uparrow\downarrow}(\mathbf{a})\\ \Phi_{\downarrow\downarrow}(\mathbf{a})\end{pmatrix}\right\}\otimes|\,\mathbf{x}\rangle$$

$$|\,\Psi^{RB}(\mathbf{k},\mathbf{s})\rangle_I = \sum_{\mathbf{x}\in\mathscr{D}_I}\left\{A^{RB}_{I,I}e^{i\langle\mathbf{k}|\mathbf{x}\rangle}\begin{pmatrix}\Phi_{\uparrow\uparrow}(\mathbf{a})\\ \Phi_{\uparrow\downarrow}(\mathbf{a})\\ \Phi_{\downarrow\uparrow}(\mathbf{a})\\ \Phi_{\downarrow\downarrow}(\mathbf{a})\end{pmatrix} + A^{RB}_{\Sigma,I}e^{i\langle\Sigma\mathbf{k}|\mathbf{x}\rangle}\begin{pmatrix}\Phi_{\uparrow\uparrow}(\mathbf{a})\\ \Phi_{\downarrow\uparrow}(\mathbf{a})\\ \Phi_{\uparrow\downarrow}(\mathbf{a})\\ \Phi_{\downarrow\downarrow}(\mathbf{a})\end{pmatrix}\right\}\otimes|\,\mathbf{x}\rangle$$

Connection with the 6 vertex model

$$\mathsf{R}(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(u) & b(u) & 0 \\ 0 & b(u) & a(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$a(u) = \frac{T^+(u) - T^-(u)}{2}$$

$$b(u) = \frac{T^+(u) + T^-(u)}{2}$$

$$T^{\pm} = \frac{\Phi^{1,2}_{\downarrow\uparrow} e^{-i\chi} \pm \Phi^{1,2}_{\uparrow\downarrow}}{\Phi^{1,2}_{\uparrow\downarrow} e^{-i\chi} \pm \Phi^{1,2}_{\downarrow\uparrow}}$$

$\Phi^{1,2}$ depend on v_1, v_2

$$v_i = \partial_{k_i} \omega_{s_i}(k_i)$$

Connection with the 6 vertex model

$$\text{Define } v_i = \tanh\theta_i \quad , \quad e^{u_{ij}} = \frac{s_i e^{\theta_i}}{s_j e^{\theta_j}} \quad , \quad \gamma = i\chi$$

$$\Rightarrow \quad T_{ij}^{\pm} = \frac{e^{-\gamma} e^{u_{ij}} \pm 1}{e^{-\gamma} \pm e^{u_{ij}}} \quad \Rightarrow \quad \begin{cases} T_{ij}^{+} = \dfrac{\cosh\left(\frac{u_{ij}-\gamma}{2}\right)}{\cosh\left(\frac{u_{ij}+\gamma}{2}\right)} \\ T_{ij}^{-} = -\dfrac{\sinh\left(\frac{u_{ij}-\gamma}{2}\right)}{\sinh\left(\frac{u_{ij}+\gamma}{2}\right)} \end{cases}$$

Connection with the 6 vertex model

$$\begin{cases} T^+_{ij} = \dfrac{\cosh\left(\frac{u_{ij}-\gamma}{2}\right)}{\cosh\left(\frac{u_{ij}+\gamma}{2}\right)} \\ T^-_{ij} = -\dfrac{\sinh\left(\frac{u_{ij}-\gamma}{2}\right)}{\sinh\left(\frac{u_{ij}+\gamma}{2}\right)} \end{cases} \Rightarrow \begin{aligned} a(u) &= \frac{T^+(u) - T^-(u)}{2} = \frac{\sinh u}{\sinh(u+\gamma)} \\ b(u) &= \frac{T^+(u) + T^-(u)}{2} = \frac{\sinh \gamma}{\sinh(u+\gamma)} \end{aligned}$$

$$\Rightarrow \quad \mathsf{R}(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(u) & b(u) & 0 \\ 0 & b(u) & a(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{\sinh u}{\sinh(u+\gamma)} & \dfrac{\sinh \gamma}{\sinh(u+\gamma)} & 0 \\ 0 & \dfrac{\sinh \gamma}{\sinh(u+\gamma)} & \dfrac{\sinh u}{\sinh(u+\gamma)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathsf{R}_{6V}(u)$$

Intertwining property: $E\, \mathsf{R}^{rel}_{ij} = \mathsf{R}^{abs}_{ij}\, E \quad ; \quad E^\dagger E = \mathbb{1}$

$$E\, \mathsf{R}^{rel}_{12}\, \mathsf{R}^{rel}_{13}\, \mathsf{R}^{rel}_{23} = \mathsf{R}^{abs}_{12}\, E\, \mathsf{R}^{rel}_{13}\, \mathsf{R}^{rel}_{23} = \cdots = \mathsf{R}^{abs}_{12}\, \mathsf{R}^{abs}_{13}\, \mathsf{R}^{abs}_{23}\, E$$

Yang-Baxter $\|$

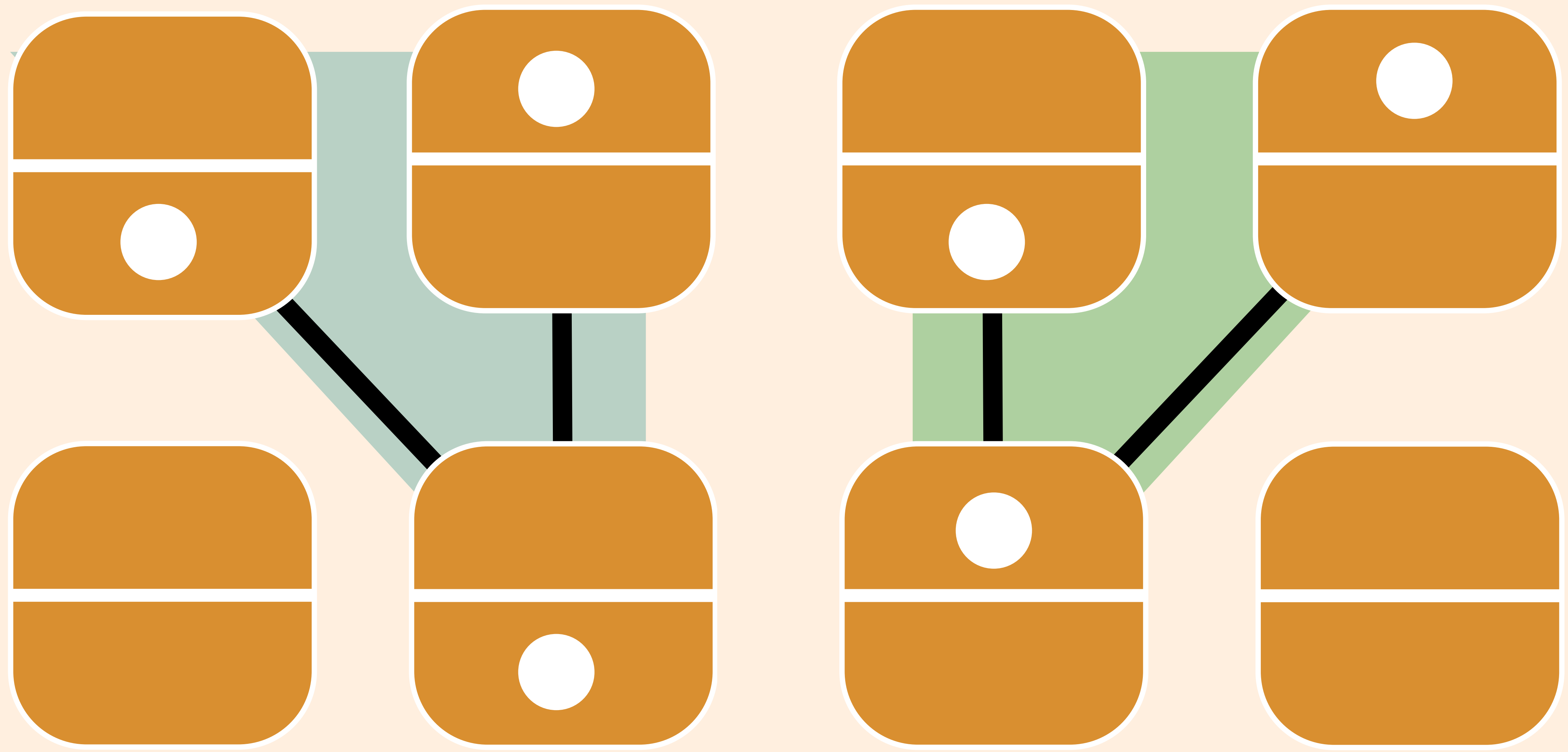
$$E\, \mathsf{R}^{rel}_{23}\, \mathsf{R}^{rel}_{13}\, \mathsf{R}^{rel}_{12} = \cdots = \mathsf{R}^{abs}_{23}\, \mathsf{R}^{abs}_{13}\, \mathsf{R}^{abs}_{12}\, E$$

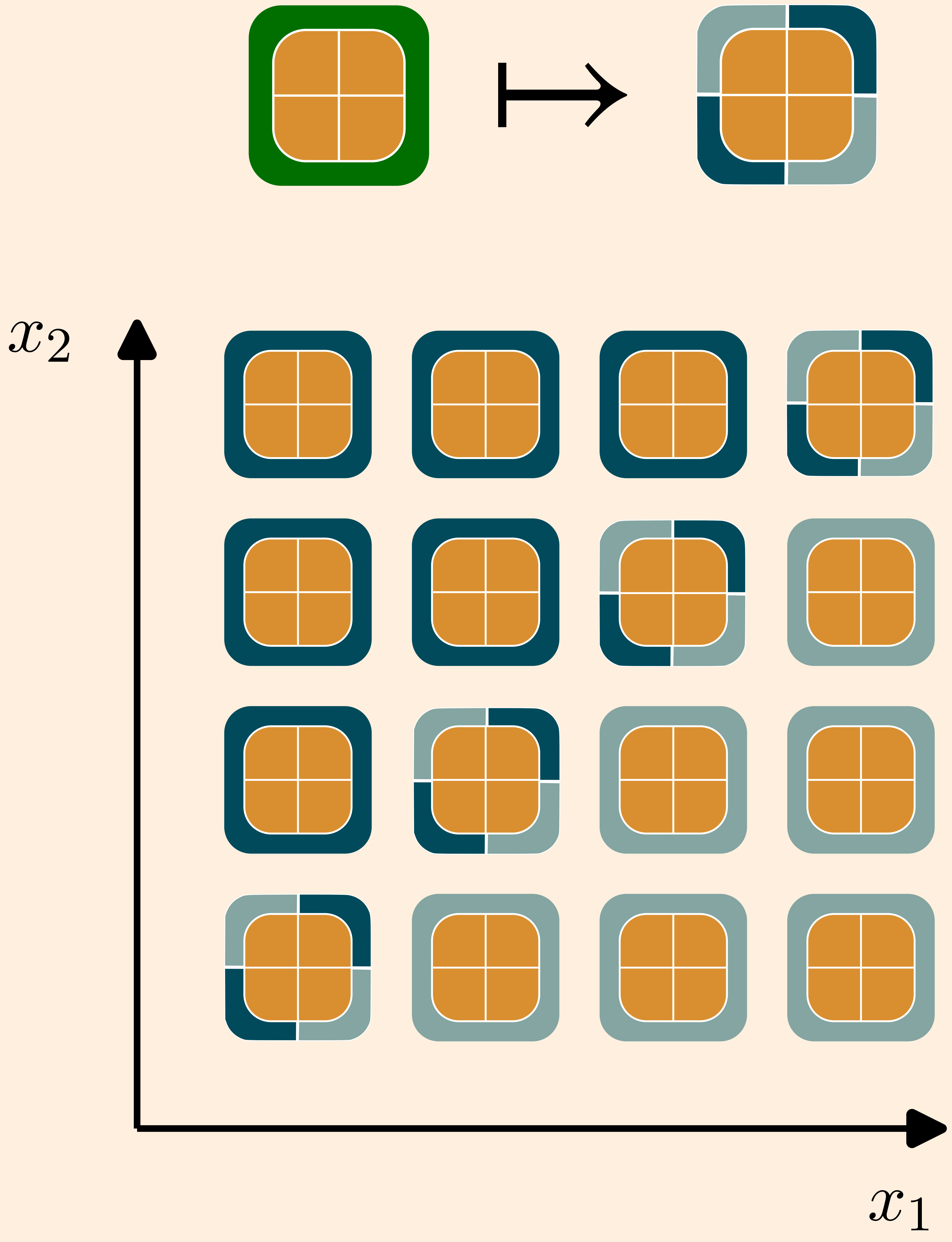
$$E^\dagger E\, \mathsf{R}^{rel}_{12}\, \mathsf{R}^{rel}_{13}\, \mathsf{R}^{rel}_{23} = E^\dagger E\, \mathsf{R}^{rel}_{23}\, \mathsf{R}^{rel}_{13}\, \mathsf{R}^{rel}_{12}$$

$$W = \begin{bmatrix} nT & im\square \\ im\square & nT^{\dagger} \end{bmatrix}$$

$x_1 < x_2$

$x_1 < x_2$





$$|\Psi(\mathbf{k},\mathbf{s})\rangle = |\Psi\rangle_I + |\Psi\rangle_\Sigma$$