

High-precision Monte Carlo predictions for radiative return with BabaYaga@NLO

End of Year Seminar

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Supervisors: Prof. Fulvio Piccinini
Prof. Guido Montagna

September 16, 2026



UNIVERSITÀ
DI PAVIA

The Muon $g - 2$



The Muon anomaly

$$\vec{\mu} = -g\mu_B\vec{S} \quad a_\mu = \frac{(g-2)_\mu}{2}$$



Muon $g-2$

In the SM we can write:

$$a_\mu = \underbrace{a_\mu^{\text{QED}}}_{>99.99\%} + a_\mu^{\text{EW}} + \underbrace{a_\mu^{\text{had}}}_{\text{Non-perturbative}}$$

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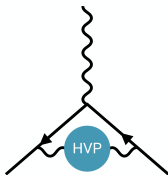
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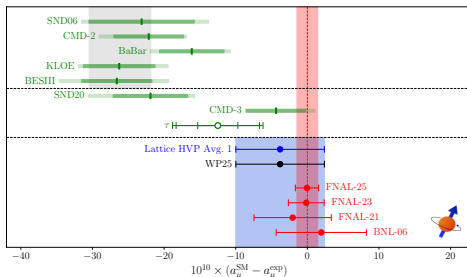
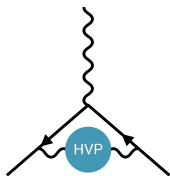
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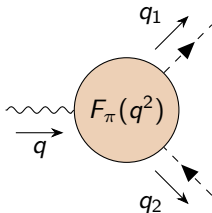
$e^+e^- \rightarrow \pi^+\pi^-$ channel

$$a_\mu^{\text{HVP-LO}} = \left(\frac{\alpha m_\mu}{3\pi} \right)^2 \int_{m_{\pi_0}^2}^{\infty} ds \frac{1}{s^2} \mathcal{R}_0^{\text{had}}(s) \hat{\mathcal{K}}(s)$$

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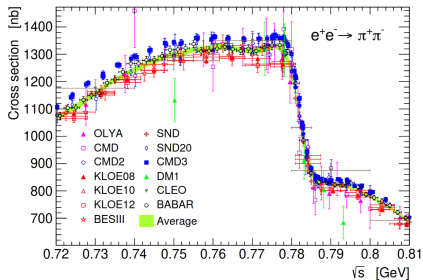
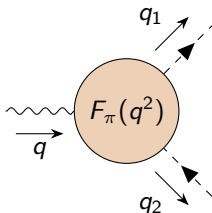
$$\langle \pi^+(q_2) \pi^-(q_1) | J_{\pi}^{\mu}(0) | 0 \rangle = e(q_1 - q_2)^{\mu} F_{\pi}(q^2)$$



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The radiative return



At fixed collider energy $\sqrt{s}$, lower-energy two pion pairs can be studied via initial-state radiation:

$$e^+e^- \rightarrow \pi^+\pi^-\gamma, \quad M_{\pi\pi}^2 = (p_{\pi^+} + p_{\pi^-})^2 < s$$

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The measured differential cross section relates to the pion production cross section:

$$\frac{d\sigma_{\pi^+\pi^-\gamma}}{dM_{\pi\pi}} = H(s, M_{\pi\pi}^2) \sigma_{\pi^+\pi^-}(M_{\pi\pi}^2),$$

where $H(s, M_{\pi\pi}^2)$ is the radiator function.

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A precise theoretical treatment of radiative corrections is required to extract the pion form factor at the sub-percent level.

The perturbative accuracy



LL resummation

Approximate calculation of
soft-virtual corrections to any order.

$$1 \quad \text{---}\rightarrow\text{---}$$

$$c_{\alpha}^{\text{LL}} \quad \text{---}\overset{\text{curly}}{\curvearrowright}\text{---} \quad \text{---}\overset{\text{curly}}{\curvearrowleft}\text{---}\rightarrow\text{---}$$

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NLO

Exact calculation up to $\mathcal{O}(\alpha)$ w.r.t. the born scattering amplitude.

- Virtual corrections



- Real corrections



The perturbative accuracy



LL resummation

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$$\begin{array}{ccc}
 \alpha^0 & & \\
 \alpha L & \alpha & \\
 \frac{1}{2}\alpha^2 L^2 & \frac{1}{2}\alpha^2 L & \frac{1}{2}\alpha^2 \\
 \sum_{n=3}^{\infty} \frac{\alpha^n}{n!} L^n & \sum_{n=3}^{\infty} \frac{\alpha^n}{n!} L^{n-1} & \dots
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NLO+LL (NLOPS)

LO	α^0		
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NNLO	$\frac{1}{2}\alpha^2 L^2$	$\frac{1}{2}\alpha^2 L$	$\frac{1}{2}\alpha^2$
h.o.	$\sum_{n=3}^{\infty} \frac{\alpha^n}{n!} L^n$	$\sum_{n=3}^{\infty} \frac{\alpha^n}{n!} L^{n-1}$	$\dots$

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NLOPS master formula

$$d\sigma_{\text{NLOPS}} = \sum_{n=1}^{\infty} \exp\{-\mathcal{C}_{\alpha}^{\text{LL}}\} F_{\text{SV}} \frac{1}{n!} \left| \mathcal{M}_n^{\text{X}} \right|^2 d\Phi_n(\{p\}, \{k\}).$$

The Pion internal structure



The standard way which respects IR cancellation is $F \times \text{sQED}$:

$$\text{Diagram} = \text{Diagram} \times F_{\pi}(M_{\pi}^2)$$

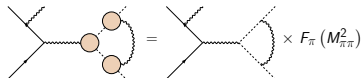
F × sQED

[2601.19530](#)



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$$= F_\pi(M_\pi^2)$$

To include the pion form factor into loop integration:

F × sQED

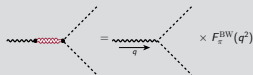
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GVMD

$$F_\pi^{\text{BW}}(q^2) = \sum_{v=1}^{n_r} F_{\pi,v}^{\text{BW}}(q^2)$$

$$F_{\pi,v}^{\text{BW}}(q^2) = \frac{c_v}{c_t} \frac{\Lambda_v^2}{\Lambda_v^2 - q^2}$$

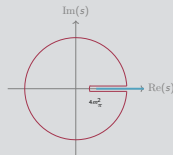


$$\times F_\pi(q^2)$$

FsQED

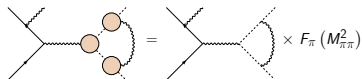
$$F_\pi(q^2) = 1 - \frac{q^2}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'} \frac{\text{Im} F_\pi(s')}{q^2 - s'}$$

$$\frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'} \text{Im} F_\pi(s') = 1$$



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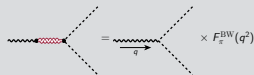


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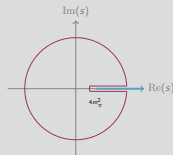
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$F \times s\text{QED}$

[2601.19530](#)



GVMD

[2603.28621](#)



FsQED

[2607.24642](#)



One Year Later



2025

- **Tuned NLO comparisons**
Completed for both $\mu^+\mu^-\gamma$
and $\pi^+\pi^-\gamma$ channels.
- **Preliminary** look at
NLOPS matching accuracy.

F. Piccinini, TI 2025

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1 Year
→

Today

- **Full NLOPS**
Achieved for $\mu^+\mu^-\gamma$ and $\pi^+\pi^-\gamma$.
- **Beyond FxsQED for pions**
 - GVMD
 - FsQED

GitHub 

Scenarios of [arxiv:2410.22882](https://arxiv.org/abs/2410.22882)

(a) KLOE I Large-Angle (LA)	(b) KLOE II Small-Angle (SA)
$\sqrt{s} = 1.02 \text{ GeV}$ $50^\circ \leq \theta^\pm \leq 130^\circ$ $p_z^\pm \geq 90 \text{ MeV} \vee p_\perp^\pm \geq 160 \text{ MeV}$ $50^\circ \leq \theta_\gamma \leq 130^\circ \wedge E_\gamma \geq 20 \text{ MeV}$ $0.1 \text{ GeV}^2 \leq M_{XX}^2 \leq 0.85 \text{ GeV}^2$	$\sqrt{s} = 1.02 \text{ GeV}$ $50^\circ \leq \theta^\pm \leq 130^\circ$ $p_z^\pm \geq 90 \text{ MeV} \vee p_\perp^\pm \geq 160 \text{ MeV}$ $\theta_{\bar{\gamma}} \leq 15^\circ \vee \theta_{\bar{\gamma}} \geq 165^\circ$ $0.35 \text{ GeV}^2 \leq M_{XX}^2 \leq 0.95 \text{ GeV}^2$
(c) BES III	(d) B
$\sqrt{s} = 4 \text{ GeV}$ $ \cos \theta^\pm \leq 0.93 \wedge p_\perp^\pm \geq 300 \text{ MeV}$ $[\cos \theta_\gamma \leq 0.8 \wedge E_\gamma \geq 25 \text{ MeV}]$ $\vee [0.86 \leq \cos \theta_\gamma \leq 0.92 \wedge E_\gamma \geq 50 \text{ MeV}]$ $\exists! \gamma : E_\gamma \geq 400 \text{ MeV}$	$\sqrt{s} = 10 \text{ GeV}$ $0.65 \text{ rad} \leq \theta^\pm \leq 2.75 \text{ rad} \wedge p^\pm \geq 1 \text{ GeV}$ $0.6 \text{ rad} \leq \theta_\gamma \leq 2.7 \text{ rad} \wedge E_\gamma \geq 3 \text{ GeV}$ $\theta_{\bar{\gamma}, \gamma^{(h)}} \leq 0.3 \text{ rad}$ $M_{XX\gamma} \geq 8 \text{ GeV}$

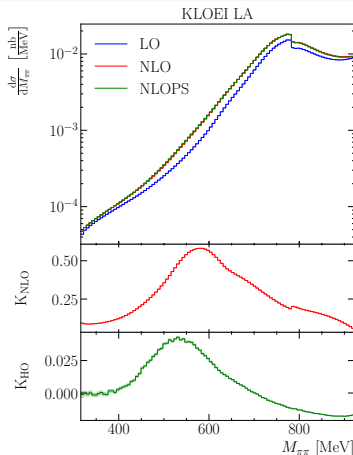
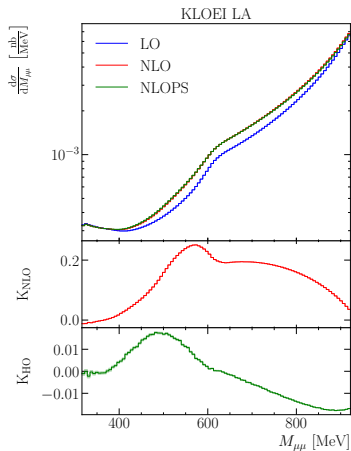
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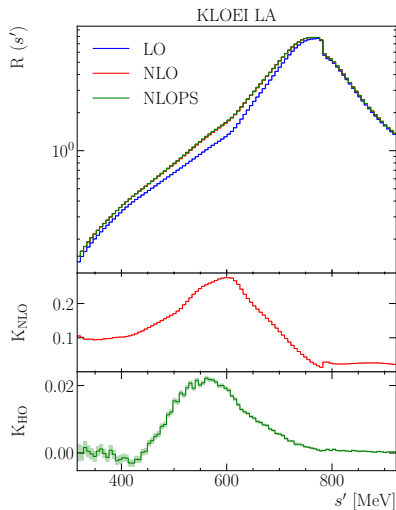
$M_{\mu\mu}, M_{\pi\pi} - F \times s\text{QED}$

$$K_{\text{NLO}} = \frac{d\sigma_{\text{NLO}} - d\sigma_{\text{LO}}}{d\sigma_{\text{LO}}}$$

$$K_{\text{HO}} = \frac{d\sigma_{\text{NLOPS}} - d\sigma_{\text{NLO}}}{d\sigma_{\text{LO}}}$$



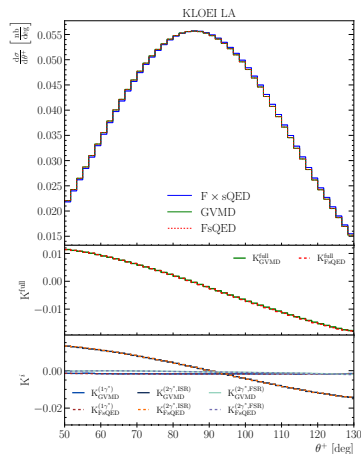
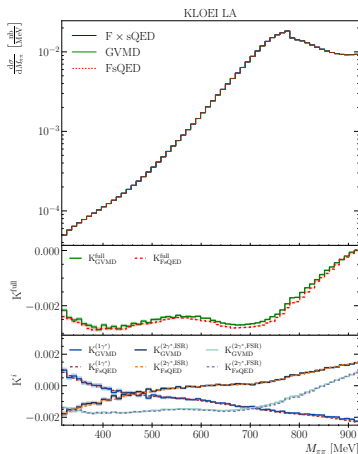
R-ratio



$$R(s') = \frac{d\sigma(e^+e^- \rightarrow \pi^+\pi^-\gamma)/ds'}{d\sigma(e^+e^- \rightarrow \mu^+\mu^-\gamma)/ds'}$$

$M_{\pi\pi}, \theta_{\pi^+}$ - Structure Dependent

$$K_i = \frac{d\sigma_i - d\sigma_{F \times sQED}}{d\sigma_{F \times sQED}} \quad i = \text{GVMD, FsQED}$$



Pion Form Factor at BESIII

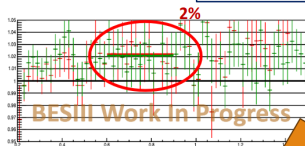
Intermediate result

2024: KKMC&AfkQED

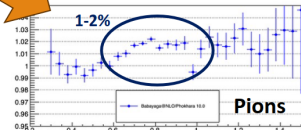
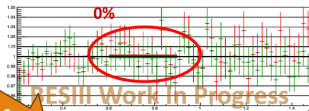
2026: BaBayaga@NLO

Status

- ✓ Selection fixed
- ✓ MC calibrations evaluated
- **HLO: MC comparison**



4C to 1C



Summary and Outlook



- Multiple photon contributions reach the **percent level** across various key observables in radiative return experiments.

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- Structure-dependent corrections are of the same order as higher-order (HO) radiative effects:
 - FsQED and GVMD results agree at the 10^{-4} level, despite GVMD theoretical limitations.
- **Future Developments**
 - $e^+e^-\gamma$ at NLOPS accuracy.
 - Extension to other hadronic channels ($K^+K^-, \dots$).
 - NNLOPS matching for ISC on $e^+e^- \rightarrow \mu^+\mu^-$ and $e^+e^- \rightarrow \gamma\gamma$.

Thank you !